Supplementary exercise on Multiple Roots

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1. The number α is called a double root of the polynomial function if $f(x) = (x - \alpha)^2 g(x)$ for some polynomial function g. Prove that α is a double root of f if and only if $f(\alpha) = 0$ and $f'(\alpha) = 0$. Hence or otherwise,
   (a) find the constants a and b so that $ax^{n+1} + bx^n + 1$ is divisible by $(x + 1)^2$, and
   (b) prove that the polynomial $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ cannot have a double root.

2 (a) The number α is called a root of multiplicity r, $r > 1$, (or an r-fold root) of a polynomial f if $f(x) = (x - \alpha)^r g(x)$ for some polynomial g such that $g(\alpha) \neq 0$. Prove the following statements:
   (i) If f has a root α of multiplicity r, then the derivative f' has a root α of multiplicity $r - 1$;
   (ii) if the derivative f' of a polynomial f has a root α of multiplicity $r - 1$ and if $f(\alpha) = 0$, then f has a root of multiplicity r .
   (b) Let n be any positive integer. Show that the polynomial $(x + 1)^n + (x - 1)^n$ has no multiple root.
   (c) Let f be a polynomial of degree 3 such that $f(x) + 1$ is divisible by $(x - 1)^2$ and $f(x) - 1$ is divisible by $(x + 1)^2$. Find the polynomial.

3. (a) Prove that $f(x)$ has α as a multiple root $\Leftrightarrow f(\alpha) = f'(\alpha) = 0$.
   (b) Let α be a double root of the polynomial $[g(x)]^2 + [h(x)]^2$, where $g(x)$ and $h(x)$ are polynomials without common factor. Show that α is also a root of $[g'(x)]^2 + [h'(x)]^2$.

4. (a) $P(x)$, $Q(x)$ are given polynomials having no common factor. Prove that the values of k for which the equation $P(x) - kQ(x) = 0$ has a multiple root are given by $k = \frac{P(\alpha)}{Q(\alpha)}$, where α is a root of the equation $P(x)\, Q'(x) - P'(x)\, Q(x) = 0$.
   (b) Hence or otherwise, find the values of k for which the equation $x^3 - 3x^2 + 3kx - 1 = 0$ has a multiple root. Solve the equation for each case.

1 (a) $a = (-1)^{n+1} n$, $b = (-1)^{n+1} (n + 1)$

2 (c) $f(x) = 0.5x^3 - 1.5x$

3 (a) (⇒) If f(x) has α as a multiple root,

$f(x) = (x - \alpha)^m g(x)$, where m is an integer > 1 and g(x) is a polynomial in x and $g(\alpha) \neq 0$

clearly $f(\alpha) = 0$

$f'(x) = (x - \alpha)^m g'(x) + m(x - \alpha)^{m-1} g(x)$

$f'(\alpha) = (\alpha - \alpha)^m g'(\alpha) + m(\alpha - \alpha)^{m-1} g(\alpha) = 0$

(⇐) If $f(\alpha) = f'(\alpha) = 0$

$f(x) = (x - \alpha)^m g(x)$, where $m \in \mathbb{N}$ and g(x) is a polynomial in x and $g(\alpha) \neq 0$.

If $m = 1$, then $f'(x) = (x - \alpha) g'(x) + g(x)$

$f'(\alpha) = (\alpha - \alpha) g'(\alpha) + g(\alpha) = g(\alpha) \neq 0$ which contradict to the fact that $f'(\alpha) = 0$

$\therefore m > 1$

→ f(x) has α as a multiple root

(b) Let $f(x) = [g(x)]^2 + [h(x)]^2$

$f'(x) = 2 g(x) g'(x) + 2 h(x) h'(x)$

Given α be a double root of the polynomial $[g(x)]^2 + [h(x)]^2$

By the result of (a), $f(\alpha) = f'(\alpha) = 0$

$[g(\alpha)]^2 + [h(\alpha)]^2 = 0$ ………… (1)

and $2 g(\alpha) g'(\alpha) + 2 h(\alpha) h'(\alpha) = 0$ ……… (2)

From (1) $[h(\alpha)]^2 = -[g(\alpha)]^2$

$h(\alpha) = \pm i\, g(\alpha)$ …… (3)

Sub. (3) into (2), $g(\alpha) g'(\alpha) \pm i\, g(\alpha) h'(\alpha) = 0$ …… (4)

Case 1 If $g(\alpha) = 0$, then (1) becomes $h(\alpha) = 0$

Contradict to the fact that g(x) and h(x) have no common factors.

Case 2 If $g(\alpha) \neq 0$, (4) becomes $g'(\alpha) \pm i\, h'(\alpha) = 0$

$g'(\alpha) = \mp i\, h'(\alpha)$

$[g'(\alpha)]^2 = -[h'(\alpha)]^2$

$[g'(\alpha)]^2 + [h'(\alpha)]^2 = 0$

α is also a root of $[g'(x)]^2 + [h'(x)]^2$

4 (b) $\alpha = 1, 1, -0.5 \Rightarrow k = 1, 1, -1.25$;

when $k = 1$, $x = 1, 1, 1$

when $k = -1.25$, $x = 4, -0.5, -0.5$

1. The number α is called a root of multiplicity r, $r > 1$, (or an r-fold root) of a polynomial f if $f(x) = (x-\alpha)^r g(x)$ for some polynomial g such that $g(\alpha) \neq 0$. Prove the following statements:
   (a) If f has a root α of multiplicity r, then the derivative f' has a root α of multiplicity $r-1$; (5 marks)
   (b) if the derivative f' of a polynomial f has a root α of multiplicity $r-1$ and if $f(\alpha) = 0$, then f has a root of multiplicity r; (10 marks)
   (c) if the derivative f' of a polynomial f has a root α of multiplicity $r-1$, then f does not necessarily have a root of multiplicity r. Prove it by giving a counter example. (5 marks)
2. Find the constants a and b so that $ax^{n+1} + bx^n + 1$ is divisible by $(x+1)^2$. (20 marks)
3. Prove that the polynomial $1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$ cannot have a double root. (20 marks)
4. Let f be a polynomial of degree 3 such that $f(x) + 1$ is divisible by $(x-1)^2$ and $f(x) - 1$ is divisible by $(x+1)^2$. Find the polynomial. (20 marks)
5. (a) $P(x)$, $Q(x)$ are given polynomials having no common factor. Prove that the values of k for which the equation $P(x) - kQ(x) = 0$ has a multiple root are given by $k = \frac{P(\alpha)}{Q(\alpha)}$, where α is a root of the equation $P(x)\, Q'(x) - P'(x)\, Q(x) = 0$. (6 marks)
   (b) Hence or otherwise, find the values of k for which the equation $x^3 - 3x^2 + 3kx - 1 = 0$ has a multiple root. Solve the equation for each case. (14 marks)

End of Paper

1. (a) $f(x) = (x - \alpha)^r g(x)$, where $g(x)$ is a polynomial and $g(\alpha) \neq 0$

$f'(x) = r(x - \alpha)^{r-1} g(x) + (x - \alpha)^r g'(x) = (x - \alpha)^{r-1}[r\, g(x) + (x - \alpha)\, g'(x)]$

Sub. $x = \alpha$ into $r\, g(x) + (x - \alpha)\, g'(x)$: $r\, g(\alpha) + (\alpha - \alpha)\, g'(\alpha) = r\, g(\alpha) \neq 0$

$\therefore$ f' has a root α of multiplicity $r - 1$.

(b) $f'(x) = (x - \alpha)^{r-1} Q(x)$ and $Q(\alpha) \neq 0$ and $f(\alpha) = 0$.

Let $f(x) = (x - \alpha)^m g(x)$, where $m \in \mathbb{N}$ and $g(x)$ is a polynomial in x and $g(\alpha) \neq 0$.

$f'(x) = m(x - \alpha)^{m-1} g(x) + (x - \alpha)^m g'(x) = (x - \alpha)^{m-1}[m\, g(x) + (x - \alpha)\, g'(x)]$

$\therefore (x - \alpha)^{m-1}[m\, g(x) + (x - \alpha)\, g'(x)] = (x - \alpha)^{r-1} Q(x)$

If $m > r$, dividing throughout by $(x - \alpha)^{m-r}$: $(x - \alpha)^{m-r}[m\, g(x) + (x - \alpha)\, g'(x)] = Q(x)$

Take limit as $x \to \alpha$, $\lim_{x\to\alpha} Q(x) = \lim_{x\to\alpha}(x-\alpha)[mg(x)+(x-\alpha)g'(x)] = 0$

$\Rightarrow Q(\alpha) = 0$, contradicts to the fact that $Q(\alpha) \neq 0$

If $m < r$, dividing throughout by $(x - \alpha)^{r-m}$: $m\, g(x) + (x - \alpha)\, g'(x) = (x - \alpha)^{r-m} Q(x)$

Similar contradiction aroused for $g(\alpha) = 0$.

$\therefore m = r$, f has a root of multiplicity r .

(c) Let $f(x) = x^2 - 2x$, roots = 0 or 2

$f'(x) = 2x - 2 = 2(x - 1)$; root $\alpha = 1$, multiplicity = 1

But clearly $\alpha = 1$ is not a root of $f(x) = 0$

2. Let $f(x) = ax^{n+1} + bx^n + 1$ is divisible by $(x + 1)^2$

$f'(x) = (n + 1)ax^n + nbx^{n-1}$

$f(-1) = a(-1)^{n+1} + b(-1)^n + 1 = 0$ …… (1)

$f'(-1) = (n + 1)a(-1)^n + nb(-1)^{n-1} = 0$ …… (2)

$n(1) + (2)$: $na(-1)^{n+1} + (n + 1)a(-1)^n + n = 0$

$(-1)^n a(n + 1 - n) = -n$

$a = (-1)^{n+1}n$

$(n + 1)(1) + (2)$: $(n + 1)(-1)^n b + nb(-1)^{n-1} + n + 1 = 0$

$(-1)^n b(n + 1 - n) = -(n + 1)$

$b = (-1)^{n+1}(n + 1)$

3. Let $f(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$.

If it has a common root α: $f(\alpha) = f'(\alpha) = 0$

$1 + \alpha + \frac{\alpha^2}{2!} + \cdots + \frac{\alpha^n}{n!} = 0$ …… (1)

$1 + \alpha + \frac{\alpha^2}{2!} + \cdots + \frac{\alpha^{n-1}}{(n-1)!} = 0$ …… (2)

(1) – (2): $\frac{\alpha^n}{n!} = 0 \Rightarrow \alpha = 0$

$f(0) = 1 \neq 0$

$\therefore$ it has no common root.

4. $f(x) + 1 = (x - 1)^2(ax + b)$

$f(x) = (x - 1)^2(ax + b) - 1$

$f'(x) = a(x - 1)^2 + 2(x - 1)(ax + b)$

$f(x) - 1 = (x - 1)^2(ax + b) - 2$, which is divisible by $(x + 1)^2$

$f(-1) - 1 = 0$ and $f'(-1) = 0$

$(-2)^2(-a + b) - 2 = 0$ …… (1)

$4a - 4(-a + b) = 0$ …… (2)

From (1): $-a + b = 0.5$ …… (3)

From (2): $2a - b = 0$ …… (4)

(3) + (4): $a = 0.5$; $b = 1$

$\therefore f(x) = (x - 1)^2(0.5x + 1) - 1 = (x^2 - 2x + 1)(0.5x + 1) - 1 = 0.5x^3 - 1.5x$

5. (a) The equation $P(x) - kQ(x) = 0$ has a multiple root α.

$P(\alpha) - kQ(\alpha) = 0$ …… (1)

$P'(\alpha) - kQ'(\alpha) = 0$ …… (2)

From (1) and (2): $k = \dfrac{P(\alpha)}{Q(\alpha)} = \dfrac{P'(\alpha)}{Q'(\alpha)}$

$P(\alpha)\,Q'(\alpha) - P'(\alpha)\,Q(\alpha) = 0$

$\therefore$ α is a root of the equation $P(x)\,Q'(x) - P'(x)\,Q(x) = 0$.

(b) $x^3 - 3x^2 + 3kx - 1 = 0$ has a multiple root.

$x^3 - 3x^2 - 1 + 3kx = 0$

$P(x) = x^3 - 3x^2 - 1$; $Q(x) = -3x$

$P(x)\,Q'(x) - P'(x)\,Q(x) = 0$

$(x^3 - 3x^2 - 1)(-3) - (3x^2 - 6x)(-3x) = 0$

$x^3 - 3x^2 - 1 - 3x^3 + 6x^2 = 0$

$2x^3 - 3x^2 + 1 = 0$

By testing, $x = 1$ is a root.

By division, $(x - 1)(2x^2 - x - 1) = 0$

$(x - 1)^2(2x + 1) = 0$

$x = 1$ or -0.5

$\therefore k = \dfrac{P'(\alpha)}{Q'(\alpha)} = \dfrac{3(1 - 2\times 1)}{-3} = 1$ or $\dfrac{3(0.25 + 2\times 0.5)}{-3} = -\dfrac{5}{4}$

when $k = 1$, $x^3 - 3x^2 + 3x - 1 = 0 \Rightarrow (x - 1)^3 = 0 \Rightarrow x = 1$

when $k = -\dfrac{5}{4}$, $x^3 - 3x^2 - \dfrac{15}{4}x - 1 = 0 \Rightarrow 4x^3 - 12x^2 - 15x - 4 = 0$

Put $x = -0.5$: LHS $= 4(-0.125) - 12(0.25) - 15(-0.5) - 4 = 0 =$ RHS

$\therefore 2x + 1$ is a factor.

By division, $(2x + 1)(2x^2 - 7x - 4) = 0$

$(2x + 1)(2x + 1)(x - 4) = 0$

$x = -0.5$ or 4