Maximum and Minimum Notes

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1. Method of completing the squares.

Example 1

Find the minimum and of $y = 2x^2 + 3x - 1$ and the corresponding value of x .

$$\frac{y}{2} = \left(x^2 + \frac{3}{2}x\right) - \frac{1}{2}$$

$$\frac{y}{2} = \left[x^2 + \frac{3}{2}x + \left(\frac{3}{4}\right)^2\right] - \frac{1}{2} - \left(\frac{3}{4}\right)^2$$

$$\frac{y}{2} = \left(x + \frac{3}{4}\right)^2 - \frac{17}{16}$$

$$y = 2\left(x + \frac{3}{4}\right)^2 - \frac{17}{8}$$

$$y \ge 2 \times 0^2 - \frac{17}{8} = -\frac{17}{8}$$

y is a minimum when $x = -\frac{3}{4}$; minimum value of $y = -\frac{17}{8}$

Example 2

Find the extremum of $y = -3x^2 + 12x$.

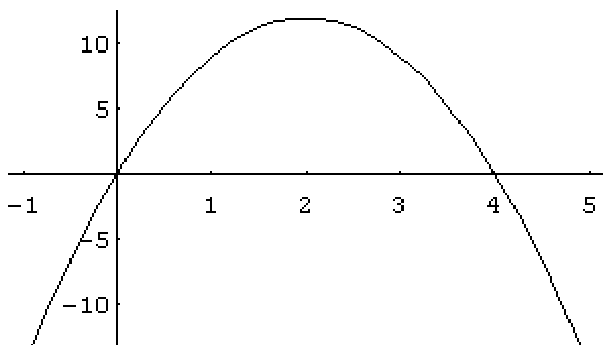
Since the coefficient of x^2 is $-12 < 0$, $\therefore$ the graph of the quadratic function opens downwards and it has a maximum only.

$$-\frac{y}{3} = x^2 - 4x$$

$$-\frac{y}{3} = x^2 - 4x \quad + 4 - 4$$

$$-\frac{y}{3} = (x-2)^2 - 4$$

$$y = -3(x-2)^2 + 12$$



From the graph, when $x = 2$, maximum $y = 12$

2. The formula $\frac{4ac-b^2}{4a}$

Let $y = ax^2 + bx + c$, $a \neq 0$.

By the method of completing the square,

$$\frac{y}{a} = \left(x^2 + \frac{b}{a}x\right) + \frac{c}{a}$$

$$\frac{y}{a} = \left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right] + \frac{c}{a} - \left(\frac{b}{2a}\right)^2$$

$$\frac{y}{a} = \left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}$$

When $a > 0$, $x = -\frac{b}{2a}$, minimum $y = \frac{4ac-b^2}{4a}$.

When $a < 0$, $x = -\frac{b}{2a}$, maximum $y = \frac{4ac-b^2}{4a}$.

Example 3

In a factory, the cost $\$C$ of producing one computer is given by $C = 5x^2 - 400x + 12000$, where x is the number of computers produced per day.

(a) Find the number of computers produced per day so that the cost of producing one computer is the least.

(b) What is the minimum cost of producing one computer?

(a) $x = -\frac{b}{2a} = -\frac{-400}{2\times 5} = 40$

40 computers should be produced per day so that the cost of producing one computer is the least.

(b) Minimum value of $C = \frac{4ac-b^2}{4a} = \frac{4\times 5\times 12000 - (-400)^2}{4\times 5} = 4000$

The minimum cost of producing one computer is \$4000.

3. Discriminant $\Delta = b^2 - 4ac$

Example 4

Let $y = 2x^2 - 2x + 1$

$2x^2 - 2x + (1 - y) = 0$, where $A = 2$, $B = -2$, $C = 1 - y$.

The equation has two real roots in x, therefore, $\Delta \ge 0$

$(-2)^2 - 4\times2\times(1 - y) \ge 0$

$y \ge \frac{1}{2}$

Minimum $y = \frac{1}{2}$, when $x = -\frac{b}{2a} = -\frac{-2}{2\times 2} = \frac{1}{2}$

Example 5

Let $y = \frac{4x-1}{x^2-2x+2}$

$yx^2 - 2yx + 2y = 4x - 1$

$yx^2 - 2(y + 2)x + (2y + 1) = 0$, where $A = y$, $B = -2(y + 2)$, $C = 2y + 1$

The equation has two real roots in x, therefore, $\Delta \ge 0$

$[-2(y + 2)]^2 - 4\times y\times(2y +1) \ge 0$

$y^2 - 3y + 4 \le 0$

$(y + 1)(y - 4) \le 0$

$-1 \le y \le 4$

Minimum $y = -1$, when $x = -\frac{B}{2A} = -\frac{-2(y+2)}{2y} = -1$

Maximum $y = 4$, when $x = -\frac{B}{2A} = -\frac{-2(y+2)}{2y} = \frac{3}{2}$

The following graph shows the above result clearly.