Lecture Notes on Partial Fractions

Reference: Techniques of Mathematics Analysis by C. J. Tranter p.15-21

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Preliminary

Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$

Let $U(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$

$$\frac{\mathrm{d}U(x)}{\mathrm{d}x} = (x-\alpha_2)\cdots(x-\alpha_n) + (x-\alpha_1)(x-\alpha_3)\cdots(x-\alpha_n) + \cdots + (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_{n-1})$$

$$= \sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}(x-\alpha_j)$$

$$\left.\frac{\mathrm{d}U(x)}{\mathrm{d}x}\right|_{x=\alpha_i} = (x-\alpha_1)\cdots(x-\alpha_{i-1})(x-\alpha_{i+1})\cdots(x-\alpha_n)\Big|_{x=\alpha_i}$$

$$= (\alpha_i-\alpha_1)\cdots(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})\cdots(\alpha_i-\alpha_n)$$

$$= \prod_{\substack{j=1\\ j\neq i}}^{n}(\alpha_i-\alpha_j)$$

In particular, if $\alpha_1 = 1, \alpha_2 = 2, \cdots, \alpha_n = n$;

$$\left.\frac{\mathrm{d}U(x)}{\mathrm{d}x}\right|_{x=\alpha_i} = (x-\alpha_1)\cdots(x-\alpha_{i-1})(x-\alpha_{i+1})\cdots(x-\alpha_n)\Big|_{x=\alpha_i}$$

$$= (i-1)(i-2)\cdots 3\cdot 2\cdot 1\cdot(-1)\cdot(-2)\cdot\cdots\cdot(i-n)$$

$$= (-1)^{n-i}(i-1)!(n-i)!$$

$$= \frac{(-1)^{n-i}(n-1)!}{C_{i-1}^{n-1}}$$

Similarly, if $\alpha_1 = -1, \alpha_2 = -2, \cdots, \alpha_n = -n$;

$$\left.\frac{\mathrm{d}U(x)}{\mathrm{d}x}\right|_{x=\alpha_i} = (x-\alpha_1)\cdots(x-\alpha_{i-1})(x-\alpha_{i+1})\cdots(x-\alpha_n)\Big|_{x=\alpha_i}$$

$$= (-i+1)(-i+2)\cdots(-3)\cdot(-2)\cdot(-1)\cdot(n-i)!$$

$$= (-1)^{i-1}(i-1)!\,(n-i)!$$

$$= \frac{(-1)^{i-1}(n-1)!}{C_{i-1}^{n-1}}$$

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Basic Skills

1. The rational function $\frac{f(x)}{g(x)}$, where f(x) and g (x) are polynomials in x is called proper if the degree of f(x) is less than the degree of g(x); it is called irreducible if f(x) and g(x) have no common factor.

2. Let $\frac{f(x)}{g(x)}$ be a rational function which is proper and irreducible. If g(x) can be factorized into relatively prime factors $p_1(x)$, p_2(x), $\cdots$, $p_r(x)$, then $\frac{f(x)}{g(x)}$ can be expressed uniquely in the form:

$\frac{q_1(x)}{p_1(x)} + \frac{q_2(x)}{p_2(x)} + \cdots + \frac{q_r(x)}{p_r(x)}$, where $\frac{q_1(x)}{p_1(x)}, \cdots , \frac{q_r(x)}{p_r(x)}$ are proper and irreducible.

3. To resolve a rational function into partial fractions, the following rules are applied:
   (a) If the degree of the numerator ≥ the degree of the denominator, divide the numerator by the denominator to obtain a proper rational function.
   (b) Factorize the denominator completely, if possible. (Note: It can be shown that every polynomial over real field can be factorized into the product of linear and quadratic factors.)
   (c) To a factor of the form $(ax + b)^n$, where n is a positive integer, there corresponds a group of partial functions:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

   (d) To a factor of the form $(ax^2 + bx + c)^n$, where n is a positive integer, there corresponds a group of partial functions:

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}.$$

Theorem 1 (Existence of Partial Fraction)

Suppose $g_1(x), \cdots, g_n(x)$ are mutually relatively prime polynomials of degree $d_1, \cdots, d_n$ respectively and $F(x)$ is a polynomial of degree less than $d_1 + \cdots + d_n$ with g.c.d.($g_i(x)$, $F(x)$) = 1 ($i = 1, \cdots, n$) .

Then there exist polynomials $\mathrm{f}_1(x), \cdots, \mathrm{f}_n(x)$ of degree less than $d_1, \cdots, d_n$ respectively such that

$$\frac{F(x)}{g_1(x)\cdots g_n(x)} \equiv \frac{\mathrm{f}_1(x)}{g_1(x)} + \cdots + \frac{\mathrm{f}_n(x)}{g_n(x)}.$$

Proof: Suffice to prove the case $n = 2$.

$\because$ $g_1(x)$, $g_2(x)$ are relatively prime polynomials.

$\therefore$ There exist polynomials $h_1(x)$, $h_2(x)$ such that $1 \equiv h_1(x)\, g_1(x) + h_2(x)\, g_2(x)$

so $F(x) \equiv F(x)\, h_1(x)\, g_1(x) + F(x)\, h_2(x)\, g_2(x)$

By division algorithm, there exists a polynomial $\mathrm{f}_2(x)$ of degree less than d_2 such that

$$F(x)\, h_1(x) \equiv g_2(x)\, q(x) + \mathrm{f}_2(x)$$

Hence $F(x) \equiv [g_2(x)\, q(x) + f_2(x)]g_1(x) + F(x)\, h_2(x)\, g_2(x)$

$$F(x) \equiv \mathrm{f}_2(x)\, g_1(x) + \mathrm{f}_1(x)\, g_2(x) \cdots\cdots (*), \text{ where } \mathrm{f}_1(x) = F(x)\, h_2(x) + q(x)\, g_1(x)$$

By formula (*), degree of L.H.S. $< d_1 + d_2$

For R.H.S., degree of $(\mathrm{f}_2(x)\, g_1(x)) < d_1 + d_2$

$\therefore$ degree of $(\mathrm{f}_1(x)\, g_2(x)) < d_1 + d_2$

But degree of $g_2(x) = d_2$

$\therefore$ degree of $\mathrm{f}_1(x) < d_1$

By (*), $F(x) \equiv \mathrm{f}_2(x)\, g_1(x) + \mathrm{f}_1(x)\, g_2(x)$ $\therefore$ $\dfrac{F(x)}{g_1(x)g_2(x)} \equiv \dfrac{\mathrm{f}_1(x)}{g_1(x)} + \dfrac{\mathrm{f}_2(x)}{g_2(x)}$

Repeat this process on $\dfrac{\mathrm{f}_1(x)}{g_1(x)}$ and $\dfrac{\mathrm{f}_2(x)}{g_2(x)}$ respectively if necessarily.

Theorem 2 (Existence of Partial Fraction)

Suppose $g(x) = P(x)^e$, where $P(x)$ is a polynomial of degree d (over $\mathbb{R}$ or $\mathbb{C}$), and $\mathrm{f}(x)$ is a polynomial of degree less than de with g.c.d.($\mathrm{f}(x)$, $P(x)$) = 1, then there exist polynomials

$s_1(x), \cdots, s_e(x)$, all of degree less than d, such that $\dfrac{\mathrm{f}(x)}{g(x)} \equiv \dfrac{s_1(x)}{P(x)} + \dfrac{s_2(x)}{P(x)^2} + \cdots + \dfrac{s_e(x)}{P(x)^e}$.

Proof: By division algorithm,

$\mathrm{f}(x) = P(x)^{e-1}\, s_1(x) + r_1(x)$, deg $r_1(x) < d(e-1)$

$r_1(x) = P(x)^{e-2}\, s_2(x) + r_2(x)$, deg $r_2(x) < d(e-2)$

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$r_{e-2}(x) = P(x)\, s_{e-1}(x) + s_e(x)$, deg $s_e(x) < d$

From the above equations, deg $s_1(x) < d$, deg $s_2(x) < d, \cdots$, deg $s_e(x) < d$

Substitute the $(e-1)$ equations back into the first equation, we have

$\mathrm{f}(x) = P(x)^{e-1}\, s_1(x) + P(x)^{e-2}\, s_2(x) + \cdots + P(x)\, s_{e-1}(x) + s_e(x)$

$$\therefore \frac{\mathrm{f}(x)}{g(x)} \equiv \frac{s_1(x)}{P(x)} + \frac{s_2(x)}{P(x)^2} + \cdots + \frac{s_e(x)}{P(x)^e}.$$

Theorem 3 (Uniqueness of Partial Fraction)

Suppose $P(x)$ and $Q(x)$ has no common factor, and deg $P(x)$ < deg $Q(x)$.

If $Q(x) = (x - \alpha_1) \cdots (x - \alpha_n)$, $\alpha_i \neq \alpha_j$ for $i \neq j$, then $\frac{P(x)}{Q(x)} \equiv \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)} \cdot \frac{1}{x-\alpha_k}$.

Proof: Induction on n. When $n = 2$,

$$\frac{P(x)}{Q(x)} \equiv \frac{P(x)}{(x-\alpha_1)(x-\alpha_2)}, \text{ where } \alpha_1 \neq \alpha_2$$

$$= \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2}, \text{ by Theorem 1}$$

$$\frac{P(x)}{Q(x)} \equiv \frac{A_1(x-\alpha_2) + A_2(x-\alpha_1)}{(x-\alpha_1)(x-\alpha_2)} \quad \ldots\ldots (1)$$

$Q(x) = (x - \alpha_1)(x - \alpha_2)$

$Q'(x) = (x - \alpha_1) + (x - \alpha_2)$

$Q'(\alpha_1) = (\alpha_1 - \alpha_2)$; $Q'(\alpha_2) = (\alpha_2 - \alpha_1)$

Compare the numerator of (1): $P(x) \equiv A_1(x - \alpha_2) + A_2(x - \alpha_1)$

$$P(\alpha_1) = A_1(\alpha_1 - \alpha_2) = A_1\, Q'(\alpha_1) \Rightarrow A_1 = \frac{P(\alpha_1)}{Q'(\alpha_1)}$$

$$P(\alpha_2) = A_2(\alpha_2 - \alpha_1) = A_2\, Q'(\alpha_2) \Rightarrow A_2 = \frac{P(\alpha_2)}{Q'(\alpha_2)}$$

$$\therefore \frac{P(x)}{Q(x)} \equiv \frac{P(\alpha_1)}{Q'(\alpha_1)} \cdot \frac{1}{x-\alpha_1} + \frac{P(\alpha_2)}{Q'(\alpha_2)} \cdot \frac{1}{x-\alpha_2}$$

Example 1 Express $\frac{5}{x^2+x-6}$ and $\frac{1}{x^2+1}$ as partial fractions.

Solution $Q(x) \equiv x^2 + x - 6$; $Q'(x) = 2x + 1$

$$\frac{5}{x^2+x-6} \equiv \frac{5}{(x+3)(x-2)}, \ \alpha_1 = -3, \alpha_2 = 2$$

$$= \frac{5}{Q'(-3)} \cdot \frac{1}{x+3} + \frac{5}{Q'(2)} \cdot \frac{1}{x-2}$$

$$= \frac{5}{2(-3)+1} \cdot \frac{1}{x+3} + \frac{5}{2(2)+1} \cdot \frac{1}{x-2}$$

$$\frac{5}{x^2+x-6} \equiv -\frac{1}{x+3} + \frac{1}{x-2}$$

$Q(x) \equiv x^2 + 1$; $Q'(x) = 2x$

$$\frac{1}{x^2+1} \equiv \frac{1}{(x+i)(x-i)}$$

$$\equiv \frac{1}{Q'(-i)} \cdot \frac{1}{x+i} + \frac{1}{Q'(i)} \cdot \frac{1}{x-i}$$

$$\equiv \frac{1}{-2i} \cdot \frac{1}{x+i} + \frac{1}{2i} \cdot \frac{1}{x-i} \Rightarrow \frac{1}{x^2+1} \equiv \frac{1}{(-2i)(x+i)} + \frac{1}{(2i)(x-i)}$$

Suppose it is true for $n = k$, i.e. $Q(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_k)$, deg $P(x) <$ deg $Q(x)$

$$\frac{P(x)}{Q(x)} \equiv \sum_{i=1}^{k} \frac{P(\alpha_i)}{Q'(\alpha_i)} \cdot \frac{1}{x-\alpha_i}$$

When $n = k + 1$, $Q(x) = (x - \alpha_1) \cdots (x - \alpha_k)(x - \alpha_{k+1})$

$$\frac{P(x)}{Q(x)} \equiv \frac{\mathrm{f}_1(x)}{Q_1(x)} + \frac{\mathrm{f}_2(x)}{x-\alpha_{k+1}} \quad \cdots\cdots (2) \text{ by theorem 1,}$$

where deg $\mathrm{f}_1(x) <$ deg $Q_1(x)$ and deg $\mathrm{f}_2(x) <$ deg $(x - \alpha_{k+1})$ and $Q_1(x) = (x - \alpha_1) \cdots (x - \alpha_k)$

By induction assumption $\frac{\mathrm{f}_1(x)}{Q_1(x)} \equiv \sum_{i=1}^{k} \frac{\mathrm{f}_1(\alpha_i)}{Q'(\alpha_i)} \cdot \frac{1}{x-\alpha_i}$

Taking common denominator of (2) on the R.H.S.

$$\frac{P(x)}{Q(x)} \equiv \frac{\mathrm{f}_1(x)(x-\alpha_{k+1}) + \mathrm{f}_2(x) Q_1(x)}{Q(x)}$$

$\because$ deg $\mathrm{f}_2(x) <$ deg$(x - \alpha_{k+1})$ $\therefore$ $\mathrm{f}_2(x) = A_{k+1}$

$\therefore$ $P(x) = \mathrm{f}_1(x)(x - \alpha_{k+1}) + A_{k+1}\, Q_1(x)$

$P(\alpha_{k+1}) = A_{k+1}\, Q_1(\alpha_{k+1}) = A_{k+1}\, Q'(\alpha_{k+1})$

$$A_{k+1} = \frac{P(\alpha_{k+1})}{Q'(\alpha_{k+1})}$$

$P(\alpha_i) = \mathrm{f}_1(\alpha_i)(\alpha_i - \alpha_{k+1})$ for $1 \le i \le k$

$$\therefore \quad \frac{\mathrm{f}_1(\alpha_i)}{Q_1'(\alpha_i)} = \frac{P(\alpha_i)}{(\alpha_i - \alpha_{k+1}) Q_1'(\alpha_i)} = \frac{P(\alpha_i)}{Q'(\alpha_i)} \quad \text{for } 1 \le i \le k$$

In other words, $\frac{P(x)}{Q(x)} \equiv \sum_{i=1}^{k+1} \frac{P(\alpha_i)}{Q'(\alpha_i)} \cdot \frac{1}{x-\alpha_i}$

Example 2 Express $\frac{1}{(x+1)(x+2)\cdots(x+n)}$ as partial fraction.

$P(x) = 1$, $Q(x) = (x + 1)(x + 2) \cdots (x + n)$

$$Q'(-i) = \frac{(-1)^{i-1}(n-1)!}{{}_{n-1}C_{i-1}},$$

$$\frac{P(x)}{Q(x)} \equiv \sum_{i=1}^{n} \frac{1}{Q'(-i)} \cdot \frac{1}{x+i} = \sum_{i=1}^{n} \frac{{}_{n-1}C_{i-1}}{(-1)^{i-1}(n-1)!} \cdot \frac{1}{x+i}$$

$$\therefore \quad \frac{1}{(x+1)(x+2)\cdots(x+n)} \equiv \frac{1}{(n-1)!}\left[\frac{{}_{n-1}C_0}{x+1} - \frac{{}_{n-1}C_1}{x+2} + \cdots + (-1)^{n-1} \cdot \frac{{}_{n-1}C_{n-1}}{x+n}\right]$$

Example 3 (Techniques of Mathematical Analysis by C J Tranter Chapter 1 p.16 Example 13)

Let $P(x)$ be a polynomial of degree n. $Q(x) \equiv (x-a_1)^2(x-a_2)\cdots(x-a_n)$, degree $n+1$,

where $a_1, a_2, \cdots, a_n$ are distinct real numbers. Express $\dfrac{P(x)}{Q(x)}$ as partial fractions.

$\because$ deg $P(x)$ < deg $Q(x)$

$$\therefore \frac{P(x)}{Q(x)} = \frac{A_0}{x-a_1} + \frac{A_1}{(x-a_1)^2} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}$$

$$\therefore P(x) \equiv A_0(x-a_1)\cdots(x-a_n) + A_1(x-a_2)\cdots(x-a_n) + \sum_{r=2}^{n} A_r(x-a_1)^2 \prod_{\substack{j=2\\ j\neq r}}^{n}(x-a_j)$$

$$P(a_1) = A_1(a_1-a_2)\cdots(a_1-a_n)$$

$$P(a_r) = A_r(a_r-a_1)^2 \prod_{\substack{j=2\\ j\neq r}}^{n}(a_r-a_j) \quad \text{for } 2 \le r \le n$$

$$P'(x) = A_0 \sum_{r=1}^{n} \prod_{\substack{j=1\\ j\neq r}}^{n}(x-a_j) + A_1 \frac{d}{dx}\left[(x-a_2)\cdots(x-a_n)\right] + \sum_{r=2}^{n} A_r \frac{\mathrm{d}}{\mathrm{d}x}\left[(x-a_1)^2 \prod_{\substack{j=2\\ j\neq r}}^{n}(x-a_j)\right]$$

$$P'(a_1) = A_0(a_1-a_2)\cdots(a_1-a_n) + A_1 \frac{\mathrm{d}}{\mathrm{d}x}\left[(x-a_2)\cdots(x-a_n)\right]\Big|_{x=a_1} + 0 \quad \cdots\cdots (1)$$

Now $\log[(x-a_2)\cdots(x-a_n)] = \log(x-a_2) + \cdots\cdots + \log(x-a_n)$

Differentiate once, $\dfrac{1}{(x-a_2)\cdots(x-a_n)} \cdot \dfrac{\mathrm{d}}{\mathrm{d}x}\left[(x-a_2)\cdots(x-a_n)\right] = \dfrac{1}{x-a_2} + \cdots + \dfrac{1}{x-a_n}$

Put $x = a_1$: $\dfrac{1}{(a_1-a_2)\cdots(a_1-a_n)} \cdot \dfrac{\mathrm{d}}{\mathrm{d}x}\left[(x-a_2)\cdots(x-a_n)\right]\Big|_{x=a_1} = \dfrac{1}{a_1-a_2} + \cdots + \dfrac{1}{a_1-a_n}$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}\left[(x-a_2)\cdots(x-a_n)\right]\Big|_{x=a_1} = (a_1-a_2)\cdots(a_1-a_n)\left(\frac{1}{a_1-a_2} + \cdots + \frac{1}{a_1-a_n}\right) \quad \cdots\cdots (2)$$

Sub. (2) into (1): $P'(a_1) = A_0(a_1-a_2)\cdots(a_1-a_n) + A_1(a_1-a_2)\cdots(a_1-a_n)\left(\dfrac{1}{a_1-a_2} + \cdots + \dfrac{1}{a_1-a_n}\right)$

$$P'(a_1) = (a_1-a_2)\cdots(a_1-a_n)\left[A_0 + A_1\left(\frac{1}{a_1-a_2} + \cdots + \frac{1}{a_1-a_n}\right)\right]$$

$$\Rightarrow A_0 = \frac{P'(a_1)}{(a_1-a_2)\cdots(a_1-a_n)} - A_1\left(\frac{1}{a_1-a_2} + \cdots + \frac{1}{a_1-a_n}\right) \quad \cdots\cdots (3)$$

$$\therefore A_r = \frac{P(a_r)}{(a_r-a_1)^2 \prod_{\substack{j=2\\ j\neq r}}^{n}(a_r-a_j)} \quad \text{for } 2 \le r \le n, \quad A_1 = \frac{P(a_1)}{(a_1-a_2)\cdots(a_1-a_n)}$$

By (3) $A_0 = \dfrac{P'(a_1)}{(a_1-a_2)\cdots(a_1-a_n)} - \dfrac{P(a_1)}{(a_1-a_2)\cdots(a_1-a_n)}\left(\dfrac{1}{a_1-a_2} + \cdots + \dfrac{1}{a_1-a_n}\right)$

$$= \frac{1}{(a_1-a_2)\cdots(a_1-a_n)}\left[P'(a_1) - P(a_1)\left(\frac{1}{a_1-a_2} + \cdots + \frac{1}{a_1-a_n}\right)\right]$$

Example 4 (Techniques of Mathematical Analysis by C J Tranter Exercise 1C p.18 Q4)

Prove that (a) $\dfrac{n!}{x(x+1)\cdots(x+n)}=\dfrac{1}{x}-\dfrac{n}{x+1}+\dfrac{\frac{1}{2}n(n-1)}{x+2}+\cdots+\dfrac{(-1)^n}{x+n}$,

(b) $\dfrac{(2n)!/n!}{x(x+1)\cdots(x+2n)}=\dfrac{1}{x(x+1)\cdots(x+n)}-\dfrac{n}{(x+1)\cdots(x+n+1)}+\dfrac{\frac{1}{2}n(n-1)}{(x+2)\cdots(x+n+2)}+\cdots+\dfrac{(-1)^n}{(x+n)\cdots(x+2n)}$

(a) Let $Q(x) = x(x + 1) \cdots (x + n)$

$$Q'(x)=\sum_{k=0}^{n}\prod_{\substack{j=0\\ j\neq k}}^{n}(x+j) \Rightarrow Q'(-k) = (-k)(-k + 1) \cdots (-1)(1)(2) \cdots (-k + n)$$

$$= (-1)^k k!\,(n-k)! = \frac{(-1)^k n!}{C_k^n}$$

$$\therefore \frac{n!}{x(x+1)\cdots(x+n)}=\sum_{k=0}^{n}\frac{n!}{Q'(-k)}\cdot\frac{1}{x+k}=\sum_{k=0}^{n}\frac{n!C_k^n}{(-1)^k n!}\cdot\frac{1}{x+k}$$

$$=\sum_{k=0}^{n}\frac{(-1)^k C_k^n}{x+k}=\frac{1}{x}-\frac{n}{x+1}+\frac{\frac{1}{2}n(n-1)}{x+2}+\cdots+\frac{(-1)^n}{x+n}$$

(b) Lemma If $0 \le k \le n$, $\displaystyle\sum_{r=0}^{k} C_{k-r}^n \cdot C_r^n = C_k^{2n}$, if $n + 1 \le k \le 2n$, $\displaystyle\sum_{r=k-n}^{n} C_{k-r}^n \cdot C_r^n = C_k^{2n}$

Proof: $(1 + x)^n(1 + x)^n = (1 + x)^{2n} \Rightarrow \left(\displaystyle\sum_{r=0}^{n} C_r^n x^r\right)\left(\displaystyle\sum_{s=0}^{n} C_s^n x^s\right)=\displaystyle\sum_{k=0}^{2n} C_k^{2n} x^k$

Compare coefficient of x^k: $0 \le k \le n$, $\displaystyle\sum_{r=0}^{k} C_{k-r}^n \cdot C_r^n = C_k^{2n}$

Compare coefficient of x^k: $n + 1 \le k \le 2n$, $\displaystyle\sum_{r=k-n}^{n} C_{k-r}^n \cdot C_r^n = C_k^{2n}$

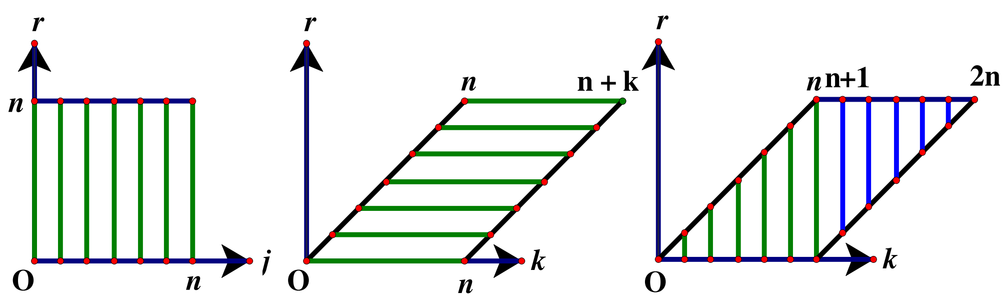
L.H.S. $=\dfrac{(2n)!/n!}{x(x+1)\cdots(x+2n)}=\dfrac{1}{n!}\displaystyle\sum_{k=0}^{2n}\dfrac{(-1)^k \cdot C_k^{2n}}{x+k}$ by the result of (a)

R.H.S. $=\displaystyle\sum_{j=0}^{n}\dfrac{(-1)^j \cdot C_j^n}{(x+j)(x+j+1)\cdots(x+j+n)}$

$=\displaystyle\sum_{j=0}^{n}\sum_{r=0}^{n}\dfrac{(-1)^j \cdot C_j^n \cdot (-1)^r \cdot C_r^n}{n!(x+r+j)}$ (replace x by $x + j$ by (a))

$=\displaystyle\sum_{r=0}^{n}\sum_{k=r}^{r+n}\dfrac{(-1)^k \cdot C_{k-r}^n \cdot C_r^n}{n!(x+k)}$ (Let $k = r + j$, when $j = 0$, $k = r$; when $j = n$, $k = r + n$)

$=\dfrac{1}{n!}\displaystyle\sum_{k=0}^{n}\sum_{r=0}^{k}\dfrac{(-1)^k \cdot C_{k-r}^n \cdot C_r^n}{(x+k)}+\dfrac{1}{n!}\displaystyle\sum_{k=n+1}^{2n}\sum_{r=k-n}^{n}\dfrac{(-1)^k \cdot C_{k-r}^n \cdot C_r^n}{(x+k)}$



$$=\frac{1}{n!}\sum_{k=0}^{n}\frac{(-1)^k}{(x+k)}\cdot\sum_{r=0}^{k}C_{k-r}^{n}\cdot C_r^n+\frac{1}{n!}\sum_{k=n+1}^{2n}\frac{(-1)^k}{(x+k)}\cdot\sum_{r=k-n}^{n}C_{k-r}^{n}\cdot C_r^n$$

$$=\frac{1}{n!}\sum_{k=0}^{n}\frac{(-1)^k\cdot C_k^{2n}}{(x+k)}+\frac{1}{n!}\sum_{k=n+1}^{2n}\frac{(-1)^k\cdot C_k^{2n}}{(x+k)}\quad\text{by the lemma}$$

$$=\frac{1}{n!}\sum_{k=0}^{2n}\frac{(-1)^k C_k^{2n}}{(x+k)}=\text{LHS}$$

Example 5 (Techniques of Mathematical Analysis by C J Tranter Exercise 1C p.18 Q5)

Deduce from Example 4 (a) that $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=n-\frac{n(n-1)}{2(2)!}+\frac{n(n-1)(n-2)}{3(3)!}-\cdots+\frac{(-1)^{n-1}}{n}$.

$$\frac{n!}{x(x+1)\cdots(x+n)}=\frac{1}{x}-\frac{n}{x+1}+\frac{\frac{1}{2}n(n-1)}{x+2}+\cdots+\frac{(-1)^n}{x+n}$$

$$\frac{n}{x+1}-\frac{\frac{1}{2}n(n-1)}{x+2}+\cdots+\frac{(-1)^{n-1}}{x+n}=\frac{1}{x}-\frac{n!}{x(x+1)\cdots(x+n)}$$

$$=\frac{(x+1)\cdots(x+n)-n!}{x(x+1)\cdots(x+n)}$$

Numerator of R.H.S. must be in the form $x^n + a_{n-1}x^{n-1} + \cdots + a_1x$.

After cancelling the common factor x in R.H.S., numerator of R.H.S. = $x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1$

Put $x = 0$ into both sides: $n-\frac{\frac{1}{2}n(n-1)}{2}+\frac{\frac{1}{3!}n(n-1)(n-2)}{2}-\cdots+\frac{(-1)^{n-1}}{n}=\frac{a_1}{n!}$

a_1 = coefficient of x in $(x + 1)\cdots(x + n) - n!$

$= 2\times3\times\cdots\times n + 1\times3\times\cdots\times n + 1\times2\times4\times\cdots\times n + 1\times2\times\cdots\times(n-1)$

$=n!\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)$

$\therefore\ 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=n-\frac{n(n-1)}{2(2)!}+\frac{n(n-1)(n-2)}{3(3)!}-\cdots+\frac{(-1)^{n-1}}{n}$.

1. Resolve $\frac{1}{(1+x)(1+2x)(1+3x)}$ into partial fractions.

$$\frac{1}{(1+x)(1+2x)(1+3x)}$$
$$=\frac{1}{6(x+1)\left(x+\frac{1}{2}\right)\left(x+\frac{1}{3}\right)}$$
$$=\frac{1}{6}\left[\frac{1}{\left(-1+\frac{1}{2}\right)\left(-1+\frac{1}{3}\right)(x+1)}+\frac{1}{\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}+\frac{1}{3}\right)\left(x+\frac{1}{2}\right)}+\frac{1}{\left(-\frac{1}{3}+1\right)\left(-\frac{1}{3}+\frac{1}{2}\right)\left(x+\frac{1}{3}\right)}\right]$$
$$=\frac{1}{6}\left(\frac{3}{x+1}-\frac{12}{x+\frac{1}{2}}+\frac{9}{x+\frac{1}{3}}\right)=\frac{1}{2(x+1)}-\frac{4}{2x+1}+\frac{9}{2(3x+1)}$$

2. Techniques of Mathematical Analysis by C J Tranter Exercise 1C p.18 Q1(a)
Express in partial fractions: $\frac{x}{(x-a)(x-b)}$.

2. If $a \neq b$, $\frac{x}{(x-a)(x-b)} \equiv \frac{a}{a-b}\cdot\frac{1}{x-a}+\frac{b}{b-a}\cdot\frac{1}{x-b}$

If $a = b$, let $\frac{x}{(x-a)(x-b)} \equiv \frac{C_1}{x-a}+\frac{C_2}{(x-a)^2}$

$x \equiv C_1(x-a) + C_2$, put $x = a \Rightarrow C_2 = a$

Compare coefficient of x: $C_1 = 1$

$\therefore \frac{x}{(x-a)(x-b)} \equiv \frac{1}{x-a}+\frac{a}{(x-a)^2}$

3. Express $\frac{x^3+7x^2+9x-14}{(x+3)(4-x^2)}$ as partial fractions.

$$\frac{x^3+7x^2+9x-14}{(x+3)(4-x^2)}=-\frac{x^3+7x^2+9x-14}{x^3+3x^2-4x-12}=-\left(1+\frac{4x^2+13x-2}{x^3+3x^2-4x-12}\right)$$
$$=-\left(1+\frac{4x^2+13x-2}{x^3+3x^2-4x-12}\right)=-1-\frac{4x^2+13x-2}{(x+3)(x+2)(x-2)}$$
$$=-1-\frac{4(-3)^2+13(-3)-2}{(x+3)(-3+2)(-3-2)}-\frac{4(-2)^2+13(-2)-2}{(-2+3)(x+2)(-2-2)}-\frac{4(2)^2+13(2)-2}{(2+3)(2+2)(x-2)}$$
$$=-1+\frac{1}{x+3}-\frac{3}{x+2}-\frac{2}{x-2}$$

4. Advanced Level Pure Mathematics S. L. Green p.330 Example 11
Express the following irreducible fraction into partial fraction: $\frac{px^2+qx+r}{(x-a)(x-b)^2}$, where $a \neq b$.

4. $\frac{px^2+qx+r}{(x-a)(x-b)^2} \equiv \frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{(x-b)^2}$

$px^2 + qx + r \equiv A(x-b)^2 + B(x-a)(x-b) + C(x-a)$

Put $x = a \Rightarrow A = \frac{pa^2+qa+r}{(a-b)^2}$

Put $x = b \Rightarrow C = \frac{pb^2+qb+r}{b-a}$

Compare coefficient of x^2: $p = A + B \Rightarrow B = p - \dfrac{pa^2+qa+r}{(a-b)^2} = \dfrac{pb^2-qa-2abp-r}{(a-b)^2}$

$$\frac{px^2+qx+r}{(x-a)(x-b)^2} \equiv \frac{pa^2+qa+r}{(a-b)^2}\cdot\frac{1}{x-a} + \frac{pb^2-qa-2abp-r}{(a-b)^2}\cdot\frac{1}{x-b} + \frac{pb^2+qb+r}{b-a}\cdot\frac{1}{(x-b)^2}$$

5. Advanced Level Pure Mathematics S. L. Green p.335 Q14

Express $\dfrac{1}{x(x+2)}$ in partial fractions. Hence find the sum of n terms of the series

$$\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \cdots$$

5. $\dfrac{1}{x(x+2)} \equiv \dfrac{1}{2x} - \dfrac{1}{2(x+2)}$

$$\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \cdots = \frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)+\cdots\right]$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(1-\frac{1}{2n+1}\right) = \frac{1}{2}$$

6. Mastering A. L. Pure Mathematics Volume 1 p.177 Q15

(a) If $a \neq b$, prove that, in general $\dfrac{\sin x}{\sin(x-a)\sin(x-b)}$ can be expressed in the form $\dfrac{A}{\sin(x-a)} + \dfrac{B}{\sin(x-b)}$, where A and B are trigonometric functions independent of x.

(b) Extend the above result to $\dfrac{\sin^2 x}{\sin(x-a)\sin(x-b)\sin(x-c)}$ for distinct numbers a, b, c.

6. (a) $\dfrac{\sin x}{\sin(x-a)\sin(x-b)} \equiv \dfrac{A}{\sin(x-a)} + \dfrac{B}{\sin(x-b)}$

$\sin x \equiv A \sin(x-b) + B \sin(x-a)$

Put $x = a \Rightarrow A = \dfrac{\sin a}{\sin(a-b)}$; put $x = b \Rightarrow B = \dfrac{\sin b}{\sin(b-a)}$

$$\frac{\sin x}{\sin(x-a)\sin(x-b)} \equiv \frac{\sin a}{\sin(a-b)\sin(x-a)} + \frac{\sin b}{\sin(b-a)\sin(x-b)}$$

(b) $\dfrac{\sin^2 x}{\sin(x-a)\sin(x-b)\sin(x-c)} \equiv \dfrac{A}{\sin(x-a)} + \dfrac{B}{\sin(x-b)} + \dfrac{C}{\sin(x-c)}$

$\sin^2 x \equiv A \sin(x-b)\sin(x-c) + B \sin(x-a)\sin(x-c) + C \sin(x-a)\sin(x-b)$

Put $x = a \Rightarrow A = \dfrac{\sin^2 a}{\sin(a-b)\sin(a-c)}$, $B = \dfrac{\sin^2 b}{\sin(b-a)\sin(b-c)}$, $C = \dfrac{\sin^2 c}{\sin(c-a)\sin(c-b)}$

$$\frac{\sin^2 x}{\sin(x-a)\sin(x-b)\sin(x-c)}$$

$$\equiv \frac{\sin^2 a}{\sin(a-b)\sin(a-c)\sin(x-a)} + \frac{\sin^2 b}{\sin(b-a)\sin(b-c)\sin(x-b)} + \frac{\sin^2 c}{\sin(c-a)\sin(c-b)\sin(x-c)}$$

7. If $\dfrac{x^2+1}{(1+x)(1+2x)\cdots(1+nx)}\equiv\sum_{k=1}^{n}\dfrac{A_k}{1+kx}$ for all value of $n \ge 3$, find A_k.

7. $$\frac{x^2+1}{(1+x)(1+2x)\cdots(1+nx)}=\frac{1}{n!}\cdot\frac{x^2+1}{(x+1)\left(x+\frac{1}{2}\right)\cdots\left(x+\frac{1}{n}\right)}$$

$$=\frac{1}{n!}\sum_{k=1}^{n}\frac{\left(-\frac{1}{k}\right)^2+1}{Q'\left(-\frac{1}{k}\right)\left(x+\frac{1}{k}\right)}\text{, where } Q(x)=(x+1)\left(x+\tfrac{1}{2}\right)\cdots\left(x+\tfrac{1}{n}\right)$$

$$=\frac{1}{n!}\sum_{k=1}^{n}\frac{1+k^2}{k(1+kx)}\cdot\frac{1}{\prod_{\substack{r=1\\r\ne k}}^{n}\left(-\frac{1}{k}+\frac{1}{r}\right)}$$

$$=\frac{1}{n!}\sum_{k=1}^{n}\frac{1+k^2}{k(1+kx)}\cdot\prod_{\substack{r=1\\r\ne k}}^{n}\left(\frac{kr}{k-r}\right)$$

$$=\frac{1}{n!}\sum_{k=1}^{n}\frac{1+k^2}{k(1+kx)}\cdot\frac{\frac{n!}{k}k^{n-1}}{(k-1)!(-1)^{n-k}(n-k)!}$$

$$=\sum_{k=1}^{n}\frac{(-1)^{n-k}k^{n-2}}{n!}\cdot\frac{n!}{k!(n-k)!}\cdot\frac{1+k^2}{(1+kx)}$$

$$=\sum_{k=1}^{n}\frac{(-1)^{n-k}k^{n-2}C_k^n\left(1+k^2\right)}{n!}\cdot\frac{1}{(1+kx)}\text{, } A_k=\frac{(-1)^{n-k}k^{n-2}C_k^n\left(1+k^2\right)}{n!}$$

8. Aids to Advanced Level Pure Mathematics Part 2 p.67 Q1
Resolve into partial fractions:

(a) $\dfrac{x^2-x+1}{x^3-x^2-x+1}$ (b) $\dfrac{5x^3+80x-77x+163}{\left(x^2+2\right)(x+7)(2x-5)}$

(c) $\dfrac{4x^3-3x^2+8x-2}{\left(x^2+1+1\right)\left(2x^2-x+3\right)}$ (d) $\dfrac{20x^2+34x+8}{(x+2)^2\left(x^3+2x^2-2x-4\right)}$

(e) $\dfrac{1}{(1-x)^n(2-x)^2}$ (f) $\dfrac{x}{(x-1)(2x-1)(x-2)^n}$

9. Aids to Advanced Level Pure Mathematics Part 2 p.68 Q2
Express in partial fractions: $\dfrac{x^3}{(x-a)(x-b)(x-c)}$. Hence prove that

$$\frac{a^3}{(a-b)(a-c)(a-d)}+\frac{b^3}{(b-c)(b-d)(b-a)}+\frac{c^3}{(c-d)(c-a)(c-b)}+\frac{d^3}{(d-a)(d-b)(d-c)}\equiv 1.$$

9. $$\frac{x^3}{(x-a)(x-b)(x-c)}\equiv\frac{x^3-(x-a)(x-b)(x-c)}{(x-a)(x-b)(x-c)}+1\equiv\frac{(a+b+c)x^2-(ab+bc+ca)x+abc}{(x-a)(x-b)(x-c)}+1$$

Let $f(x) = (a + b + c)x^2 - (ab + bc + ca)x + abc$; $g(x) = (x - a)(x - b)(x - c)$
$g'(x) = (x - b)(x - c) + (x - a)(x - c) + (x - a)(x - b)$
Case 1 If a, b and c are all distinct

$$\frac{f(a)}{g'(a)}=\frac{(a+b+c)a^2-(ab+bc+ca)a+abc}{(a-b)(a-c)}=\frac{a^3}{(a-b)(a-c)}$$

Similarly $\frac{f(b)}{g'(b)}=\frac{b^3}{(b-a)(b-c)}$, $\frac{f(c)}{g'(c)}=\frac{c^3}{(c-a)(c-b)}$

$\therefore\ \frac{x^3}{(x-a)(x-b)(x-c)}\equiv\frac{a^3}{(a-b)(a-c)}\cdot\frac{1}{x-a}+\frac{b^3}{(b-a)(b-c)}\cdot\frac{1}{x-b}+\frac{c^3}{(c-a)(c-b)}\cdot\frac{1}{x-c}+1$

Case 2 If $a = b = c$, let $\frac{f(x)}{g(x)}\equiv\frac{d_1}{x-a}+\frac{d_2}{(x-a)^2}+\frac{d_3}{(x-a)^3}$

$f(x) = 3ax^2 - 3a^2x + a^3 \equiv d_1(x-a)^2 + d_2(x-a) + d_3$

$d_3 = f(a) = 3a^3 - 3a^3 + a^3 = a^3$

Differentiate w.r.t. x and put $x = a$: $d_2 = 6a^2 - 3a^2 = 3a^2$

Compare coefficient of x^2: $d_1 = 3a$

$\therefore\ \frac{x^3}{(x-a)^3}\equiv\frac{3a}{x-a}+\frac{3a^2}{(x-a)^2}+\frac{a^3}{(x-a)^3}+1$

Case 3 If two are equal, the third is different. WLOG assume $a = b \neq c$

$\frac{f(x)}{g(x)}\equiv\frac{e_1}{x-a}+\frac{e_2}{(x-a)^2}+\frac{k}{x-c}$

Compare the numerator: $f(x) = (2a+c)x^2 - (a^2+2ac)x + a^2c \equiv e_1(x-a)(x-c) + e_2(x-c) + k(x-a)^2$

Put $x = c$, $(2a+c)c^2 - (a^2+2ac)c + a^2c \equiv k(c-a)^2 \Rightarrow k=\frac{c^3}{(c-a)^2}$

Put $x = a$, $(2a+c)a^2 - (a^2+2ac)a + a^2c \equiv e_2(a-c) \Rightarrow e_2=\frac{a^3}{a-c}$

Differentiate and put $x = a$: $2(2a+c)a - (a^2+2ac) \equiv e_1(a-c) + e_2 \Rightarrow e_1=\frac{2a^3-3a^2c}{(a-c)^2}$

$\therefore\ \frac{x^3}{(x-a)^2(x-c)}\equiv\frac{2a^3-3a^2c}{(a-c)^2(x-a)}+\frac{a^3}{(a-c)(x-a)^2}+\frac{c^3}{(a-c)^2(x-c)}+1$

To prove that $\frac{a^3}{(a-b)(a-c)(a-d)}+\frac{b^3}{(b-c)(b-d)(b-a)}+\frac{c^3}{(c-d)(c-a)(c-b)}+\frac{d^3}{(d-a)(d-b)(d-c)}\equiv 1$

where a, b, c, d are distinct.

Put $x = d$ in $\frac{x^3}{(x-a)(x-b)(x-c)}\equiv\frac{a^3}{(a-b)(a-c)}\ \frac{1}{x-a}+\frac{b^3}{(b-a)(b-c)}\ \frac{1}{x-b}+\frac{c^3}{(c-a)(c-b)}\ \frac{1}{x-c}+1$

$\frac{d^3}{(d-a)(d-b)(d-c)}\equiv\frac{a^3}{(a-b)(a-c)}\ \frac{1}{d-a}+\frac{b^3}{(b-a)(b-c)}\ \frac{1}{d-b}+\frac{c^3}{(c-a)(c-b)}\ \frac{1}{d-c}+1$

$\therefore\frac{a^3}{(a-b)(a-c)(a-d)}+\frac{b^3}{(b-c)(b-d)(b-a)}+\frac{c^3}{(c-d)(c-a)(c-b)}+\frac{d^3}{(d-a)(d-b)(d-c)}\equiv 1$

10. Techniques of Mathematical Analysis by C J Tranter Exercise 1C p.18 Q2

Express in partial fractions: $\dfrac{(x-a)(x-b)(x-c)(x-d)}{(x+a)(x+b)(x+c)(x+d)}$

(a) when a, b, c, d are all unequal,

(b) when they are all equal.

10. $\dfrac{(x-a)(x-b)(x-c)(x-d)}{(x+a)(x+b)(x+c)(x+d)} \equiv \dfrac{(x-a)(x-b)(x-c)(x-d)-(x+a)(x+b)(x+c)(x+d)}{(x+a)(x+b)(x+c)(x+d)}+1$

Let $f(x) = (x-a)(x-b)(x-c)(x-d) - (x+a)(x+b)(x+c)(x+d)$

$g(x) = (x+a)(x+b)(x+c)(x+d)$

(a) When a, b, c, d are all unequal.

$f(-a) = 2a(a+b)(a+c)(a+d)$, $g'(-a) = (b-a)(c-a)(d-a)$

$$\frac{(x-a)(x-b)(x-c)(x-d)}{(x+a)(x+b)(x+c)(x+d)} \equiv \frac{2a(a+b)(a+c)(a+d)}{(b-a)(c-a)(d-a)(x+a)} + \frac{2b(b+a)(b+c)(b+d)}{(a-b)(c-b)(d-b)(x+b)}$$
$$+\frac{2c(c+a)(c+b)(c+d)}{(a-c)(b-c)(d-c)(x+c)} + \frac{2d(d+a)(d+b)(d+c)}{(a-d)(b-d)(c-d)(x+d)} + 1$$

(b) If $a = b = c = d$, $\dfrac{f(x)}{g(x)} \equiv \dfrac{e_1}{x+a} + \dfrac{e_2}{(x+a)^2} + \dfrac{e_3}{(x+a)^3} + \dfrac{e_4}{(x+a)^4}$

$f(x) = (x-a)^4 - (x+a)^4 = e_1(x+a)^3 + e_2(x+a)^2 + e_3(x+a) + e_4$

$e_4 = f(-a) = 16a^4$

$f'(x) = 4(x-a)^3 - 4(x+a)^3 = 3e_1(x+a)^2 + 2e_2(x+a) + e_3$

$e_3 = f'(-a) = -32a^3$

$f''(x) = 12(x-a)^2 - 12(x+a)^2 = 6e_1(x+a) + 2e_2$

$e_2 = \dfrac{1}{2}f''(-a) = 24a^2$

$f'''(x) = 24(x-a) - 24(x+a) = 6e_1 \Rightarrow e_1 = -8a$

$\therefore\ \dfrac{(x-a)^4}{(x+a)^4} \equiv -\dfrac{8a}{x+a} + \dfrac{24a^2}{(x+a)^2} - \dfrac{32a^3}{(x+a)^3} + \dfrac{16a^4}{(x+a)^4} + 1$

11. Techniques of Mathematical Analysis by C J Tranter Exercise 1C p.18 Q3

Evaluate $\dfrac{(a-y)(a-z)(a-u)}{(a-b)(a-c)(a-d)(a-x)} + \dfrac{(b-y)(b-z)(b-u)}{(b-a)(b-c)(b-d)(b-x)} + \dfrac{(c-y)(c-z)(c-u)}{(c-a)(c-b)(c-d)(c-x)} + \dfrac{(d-y)(d-z)(d-u)}{(d-a)(d-b)(d-c)(d-x)}$.

11. From the expression we know that a, b, c, d are all distinct.

Consider $-\dfrac{(x-y)(x-z)(x-u)}{(x-a)(x-b)(x-c)(x-d)}$.

Let $P(x) = -(x-y)(x-z)(x-u)$, $Q(x) = (x-a)(x-b)(x-c)(x-d)$

$P(a) = -(a-y)(a-z)(a-u)$, $Q'(a) = (a-b)(a-c)(a-d)$

By Partial Fraction Theorem,

$$-\frac{(x-y)(x-z)(x-u)}{(x-a)(x-b)(x-c)(x-d)} \equiv -\frac{(a-y)(a-z)(a-u)}{(a-b)(a-c)(a-d)(x-a)} - \frac{(b-y)(b-z)(b-u)}{(b-a)(b-c)(b-d)(x-b)}$$
$$-\frac{(c-y)(c-z)(c-u)}{(c-a)(c-b)(c-d)(x-c)} - \frac{(d-y)(d-z)(d-u)}{(d-a)(d-b)(d-c)(x-d)}$$

$\therefore$ The expression $= -\dfrac{(x-y)(x-z)(x-u)}{(x-a)(x-b)(x-c)(x-d)}$.

12. Advanced Level Pure Mathematics S. L. Green p.335 Q10
Resolve into partial fractions $\frac{2}{(1-2x)^2(1+4x^2)}$ and hence obtain the coefficients of x^{4n} and x^{4n+1} in the expansion of this function in ascending powers of x. State the values of x for which the expansion is valid.

12. $\frac{2}{(1-2x)^2(1+4x^2)} \equiv \frac{A_1}{1-2x} + \frac{A_2}{(1-2x)^2} + \frac{B+Cx}{1+4x^2}$

$2 \equiv A_1(1-2x)(1+4x^2) + A_2(1+4x^2) + (B+Cx)(1-2x)^2$

Put $x = \frac{1}{2}$: $A_2 = 1$;

Put $x = \frac{1}{2i} \Rightarrow 2 = (B + \frac{C}{2i})(1+i)^2 \Rightarrow 4i = (C + 2Bi)(2i) \Rightarrow 2 = C + 2Bi \Rightarrow C = 2, B = 0$

Compare coefficient of x^3: $0 = -8A_1 + 4C \Rightarrow A_1 = \frac{1}{2}C = 1$

$\frac{2}{(1-2x)^2(1+4x^2)} \equiv \frac{1}{1-2x} + \frac{1}{(1-2x)^2} + \frac{2x}{1+4x^2}$

$\frac{1}{1-2x} + \frac{1}{(1-2x)^2} + \frac{2x}{1+4x^2} \equiv (1 + 2x + 4x^2 + \cdots + 2^n x^n + \cdots) + [1 + 2(2x) + 3(2x)^2 + \cdots + (n+1)(2x)^n + \cdots] + 2x[1 - 4x^2 + 16x^4 - \ldots + (-4x^2)^n + \cdots]$

Coefficient of $x^{4n} = 2^{4n} + (4n+1)2^{4n} = (4n+2)2^{4n}$
Coefficient of $x^{4n+1} = 2^{4n+1} + (4n+2)2^{4n+1} + 2(-4)^{2n} = (4n+4)2^{4n+1} = (n+1)2^{4n+3}$

The expansion is valid for $|x| < \frac{1}{2}$.

13. (a) Resolve $\frac{9x}{(1-x)^2(1-4x)}$ into partial fractions.

(b) Let n be a positive integer. Find the coefficient of x^n in the expansion of $\frac{9x}{(1-x)^2(1-4x)}$ for $|x| < \frac{1}{4}$.

(c) Use the result of (b) or otherwise to show that $4^{n+1} - 3n - 4$ is divisible by 9 for any positive integer n .

13. (a) $\frac{9x}{(1-x)^2(1-4x)} \equiv \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-4x}$

$9x \equiv A(1-x)(1-4x) + B(1-4x) + C(1-x)^2$

Put $x = 1$, $9 = -3B \Rightarrow B = -3$

Put $x = \frac{1}{4}$, $\frac{9}{4} = \frac{9}{16}C \Rightarrow C = 4$

Compare coefficient of x^2: $4A + C = 0 \Rightarrow A = -1$

$\therefore \frac{9x}{(1-x)^2(1-4x)} \equiv -\frac{1}{1-x} - \frac{3}{(1-x)^2} + \frac{4}{1-4x}$

(b) Expand the infinite series for $|x| < \frac{1}{4}$:

$\frac{9x}{(1-x)^2(1-4x)} = -1(1 + x + x^2 + \cdots + x^n + \cdots) - 3[1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots] + 4[1 + 4x + (4x)^2 + \cdots + (4x)^n + \cdots]$

The coefficient of x^n is $4^{n+1} - 3n - 4$.

(c) LHS $= \frac{9x}{(1-x)^2(1-4x)} = 9x[1 + 2x + 3x^2 + \cdots + (r+1)x^r + \cdots][1+4x+(4x)^2+\cdots+(4x)^n+\cdots]$

Coefficient of x^n in L.H.S. is clearly a multiple of 9 = coefficient of x^n in R.H.S.

$\therefore 4^{n+1} - 3n - 4$ is divisible by 9.

14. Mastering A. L. Pure Mathematics Volume 1 p.177 Q18

If $f(x) = (x-a)^r(x-b)^s g(x)$, where $a \neq b$, $g(a) \neq 0$, $g(b) \neq 0$, show that

$$\frac{f'(x)}{f(x)} = \frac{r}{x-a} + \frac{s}{x-b} + \frac{g'(x)}{g(x)}.$$

14. $\ln f(x) = r\ln(x-a) + s\ln(x-b) + \ln g(x)$

Differentiate w.r.t. x: $\frac{f'(x)}{f(x)} = \frac{r}{x-a} + \frac{s}{x-b} + \frac{g'(x)}{g(x)}$

15. Mastering A. L. Pure Mathematics Volume 1 p.178 Q19

If $x^n - 1 = (x-a_1)(x-a_2)\cdots(x-a_n)$, where $a_1, \cdots, a_n$ are complex numbers. Show that if $f(x)$ is a polynomial of degree < n with complex coefficients $\frac{f(x)}{x^n-1} = \frac{1}{n}\sum_{r=1}^{n}\frac{a_r f(a_r)}{x-a_r}$.

15. $$\frac{f(x)}{x^n-1} = \sum_{r=1}^{n}\frac{f(a_r)}{\frac{d}{dx}\left(x^n-1\right)\Big|_{x=a_r}\cdot(x-a_r)} = \sum_{r=1}^{n}\frac{f(a_r)}{na_r^{n-1}\cdot(x-a_r)} = \frac{1}{n}\sum_{r=1}^{n}\frac{a_r f(a_r)}{a_r^n\cdot(x-a_r)} = \frac{1}{n}\sum_{r=1}^{n}\frac{a_r f(a_r)}{x-a_r}$$

16. Aids to Advanced Level Pure Mathematics Part 2 p.68 Q3

(a) If a, b, c are distinct and non-zero numbers, resolve $E = \frac{1}{(1-ax)(1-bx)(1-cx)}$ into partial fractions.

(b) If E can be expanded as an infinite series, for what range of values of x is the expansion valid?

(c) By using (a) and (b) show that the sum of all homogenous products of degree n which can be formed from the letters a, b, c, (i.e. $a^p b^q c^r$, where p, q, r are non-negative integers such that $p+q+r=n$) is $\frac{a^{n+2}(c-b)+b^{n+2}(a-c)+c^{n+2}(b-a)}{(b-c)(c-a)(a-b)}$.

16. (a) $$E = \frac{a^2}{(a-b)(a-c)(1-ax)} + \frac{b^2}{(b-c)(b-a)(1-bx)} + \frac{c^2}{(c-a)(c-b)(1-cx)}$$

(b) If E can be expanded as an infinite series, $|x| <(\min\left(\frac{1}{|a|},\frac{1}{|b|},\frac{1}{|c|}\right)$.

(c) $$E = \frac{a^2}{(a-b)(a-c)}\left[1+ax+(ax)^2+\cdots+(ax)^n+\cdots\right] + \frac{b^2}{(b-c)(b-a)}\left[1+bx+(bx)^2+\cdots+(bx)^n+\cdots\right]$$
$$+\frac{c^2}{(c-a)(c-b)}\left[1+cx+(cx)^2+\cdots+(cx)^n+\cdots\right]$$

Coefficient of $x^n = \frac{a^{n+2}}{(a-b)(a-c)} + \frac{b^{n+2}}{(b-c)(b-a)} + \frac{c^{n+2}}{(c-a)(c-b)}$

$= \frac{a^{n+2}(c-b)+b^{n+2}(a-c)+c^{n+2}(b-a)}{(b-c)(c-a)(a-b)}$

= the sum of all homogenous products of degree n which can be formed from the letters a, b, c, (i.e. $a^p b^q c^r$, where p, q, r are non-negative integers such that $p+q+r=n$)

17. Techniques of Mathematical Analysis by C J Tranter Exercise 1(d) p.21 Q23

If a, b, c are distinct real numbers, prove that, for all values of $x \neq a$, b and c ,

$$\frac{(b-c)(c-a)(a-b)}{(x-a)(x-b)(x-c)}\left\{\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}\right\}=\frac{c-b}{(x-a)^2}+\frac{a-c}{(x-b)^2}+\frac{b-a}{(x-c)^2}$$

17. Let $f(x) = (x - a)(x - b)(x - c)$

Then $\frac{1}{(x-a)(x-b)(x-c)}=\sum\frac{1}{f'(a)(x-a)}$

$$=\frac{1}{(a-b)(a-c)(x-a)}+\frac{1}{(b-a)(b-c)(x-b)}+\frac{1}{(c-a)(c-b)(x-c)}$$

$$\Rightarrow \frac{(b-c)(c-a)(a-b)}{(x-a)(x-b)(x-c)}=\frac{c-b}{x-a}+\frac{a-c}{x-b}+\frac{b-a}{x-c}$$

$$\frac{(b-c)(c-a)(a-b)}{(x-a)(x-b)(x-c)}\left(\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}\right)$$

$$=\left(\frac{c-b}{x-a}+\frac{a-c}{x-b}+\frac{b-a}{x-c}\right)\left(\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}\right)$$

$$=\left(\frac{c-b}{x-a}+\frac{a-c}{x-b}+\frac{b-a}{x-c}\right)\left(\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}\right)$$

$$=\frac{c-b}{(x-a)^2}+\frac{a-c}{(x-b)^2}+\frac{b-a}{(x-c)^2}+\frac{(c-b)+(a-c)}{(x-a)(x-b)}+\frac{(c-b)+(b-a)}{(x-a)(x-c)}+\frac{(a-c)(b-a)}{(x-b)(x-c)}$$

$$=\frac{c-b}{(x-a)^2}+\frac{a-c}{(x-b)^2}+\frac{b-a}{(x-c)^2}+\frac{a-b}{(x-a)(x-b)}+\frac{c-a}{(x-a)(x-c)}+\frac{b-c}{(x-b)(x-c)}$$

$$=\frac{c-b}{(x-a)^2}+\frac{a-c}{(x-b)^2}+\frac{b-a}{(x-c)^2}+\left(\frac{1}{x-a}-\frac{1}{x-b}\right)+\left(\frac{1}{x-c}-\frac{1}{x-a}\right)+\left(\frac{1}{x-b}-\frac{1}{x-c}\right)$$

$$=\frac{c-b}{(x-a)^2}+\frac{a-c}{(x-b)^2}+\frac{b-a}{(x-c)^2}$$

18. Techniques of Mathematical Analysis by C J Tranter Exercise 1(d) p.21 Q22

Prove that $\frac{2^n\cdot n!}{(y+1)(y+3)\cdots(y+2n+1)}=\frac{1}{y+1}-\frac{{}_nC_1}{y+3}+\cdots+\frac{(-1)^n{}_nC_n}{y+2n+1}$.

18. $\frac{2^n\cdot n!}{(y+1)(y+3)\cdots(y+2n+1)}=\sum_{r=0}^{n}\frac{2^n\cdot n!}{Q'(-2r-1)(y+2r+1)}$, where $Q(x) = (y+1)(y+3)\cdots(y+2n+1)$

$Q'(-2r-1) = (-2r)(-2r+2)\cdots(-2)(2)(4)\ldots(-2r+2n) = (-1)^r\, 2^n\cdot r!\,(n-r)!$

$$\frac{2^n\cdot n!}{(y+1)(y+3)\cdots(y+2n+1)}=\sum_{r=0}^{n}\frac{2^n\cdot n!}{(-1)^r 2^n r!(n-r)!(y+2r+1)}=\sum_{r=0}^{n}\frac{(-1)^r C_r^n}{(y+2r+1)}$$

19. Techniques of Mathematical Analysis by C J Tranter Exercise 1(d) p.21 Q21

$f(x) = (x - \alpha_1) \cdots (x - \alpha_n)$ where .$\alpha_1, \cdots, \alpha_n$ are distinct.

(a) Show that $\sum_{r=1}^{n} \frac{\alpha_r^k}{f'(\alpha_r)} = 0$ when $k = 0, 1, \cdots, n - 2$.

(b) Show that $\sum_{r=1}^{n} \frac{\alpha_r^{n-1}}{f'(\alpha_r)} = 1$.

(c) Find the value of $\sum_{r=1}^{n} \frac{\alpha_r^n}{f'(\alpha_r)}$.

19. (a) If $k = 0, 1, 2, \cdots, n - 1$, we can use partial fractions theorem directly.

$$\frac{x^k}{f(x)} \equiv \sum_{r=1}^{n} \frac{\alpha_r^k}{f'(\alpha_r)(x-\alpha_r)}$$

Taking common denominators on R.S. and compare the numerators on both sides:

$x^k \equiv \sum_{r=1}^{n} \frac{\alpha_r^k}{f'(\alpha_r)} \prod_{\substack{s=1 \\ s \neq r}}^{n} (x-\alpha_s)$, deg(L.H.S.) = k, deg(R.H.S.) = $n - 1$

Compare coefficient of x^{n-1} : when $k = 0, 1, \cdots, n - 2$, $\sum_{r=1}^{n} \frac{\alpha_r^k}{f'(\alpha_r)} = 0$

(b) When $k = n - 1$, $\sum_{r=1}^{n} \frac{\alpha_r^{n-1}}{f'(\alpha_r)} = 1$

(c) When $k = n$, $\frac{x^n}{f(x)} \equiv 1 + \frac{x^n - f(x)}{f(x)} = 1 + \sum_{r=1}^{n} \frac{\alpha_r^n - f(\alpha_r)}{f'(\alpha_r)(x-\alpha_r)} = 1 + \sum_{r=1}^{n} \frac{\alpha_r^n}{f'(\alpha_r)(x-\alpha_r)}$

Taking common denominators on R.H.S. and compare numerators on both sides:

$$x^n \equiv f(x) + \sum_{r=1}^{n} \frac{\alpha_r^n}{f'(\alpha_r)} \prod_{\substack{s=1 \\ s \neq r}}^{n} (x-\alpha_s)$$

Compare coefficient of x^{n-1} :

0 = coefficient of x^{n-1} in $(x - \alpha_1) \cdots (x - \alpha_n) + \sum_{r=1}^{n} \frac{\alpha_r^n}{f'(\alpha_r)} \prod_{\substack{s=1 \\ s \neq r}}^{n} (x-\alpha_s)$

$$0 = -\sum_{r=1}^{n} \alpha_r + \sum_{r=1}^{n} \frac{\alpha_r^n}{f'(\alpha_r)}$$

$$\therefore \sum_{r=1}^{n} \frac{\alpha_r^n}{f'(\alpha_r)} = \sum_{r=1}^{n} \alpha_r$$

20. Techniques of Mathematical Analysis by C. J. Tranter Exercise 1(d) p.21 Q24
Given that $\phi(x) = (x - a_0)(x - a_1)\cdots(x - a_n)$, where $a_0, a_1, \cdots, a_n$ are distinct and that deg f(x) ≤ n
Show that the coefficient of x^n in f(x) is $\sum_{r=0}^{n}\frac{f(a_r)}{\phi'(a_r)}$.

20. Apply Partial faction theorem on $\frac{f(x)}{\phi(x)}$.

$$\frac{f(x)}{\phi(x)} \equiv \sum_{r=0}^{n}\frac{f(a_r)}{\phi'(a_r)(x-a_r)}$$

$f(x) \equiv \sum_{r=0}^{n}\frac{f(a_r)\phi(x)}{\phi'(a_r)(x-a_r)} = \sum_{r=0}^{n}\frac{f(a_r)}{\phi'(a_r)}\prod_{\substack{s=0\\ s\neq r}}^{n}(x-a_s)$; ∴ coefficient of x^n in f(x) is $\sum_{r=0}^{n}\frac{f(a_r)}{\phi'(a_r)}$.

21. Mastering A. L. Pure Mathematics Volume 1 p.178 Q22
If $f(x) = (x - a_1) \cdots (x - a_n)$, where $a_1, \cdots, a_n$ are distinct and that $p(x)$ is a polynomial of degree less than $n - 1$, show that $\sum_{r=1}^{n}\frac{p(a_r)}{f'(a_r)} = 0$.

21. Apply Partial faction theorem on $\frac{p(x)}{f(x)}$.

$$\frac{p(x)}{f(x)} \equiv \sum_{r=1}^{n}\frac{p(a_r)}{f'(a_r)(x-a_r)}$$

$$p(x) \equiv \sum_{r=1}^{n}\frac{p(a_r)f(x)}{f'(a_r)(x-a_r)} = \sum_{r=1}^{n}\frac{p(a_r)}{f'(a_r)}\prod_{\substack{s=1\\ s\neq r}}^{n}(x-a_s)$$

∴ coefficient of x^n in $p(x)$ is $\sum_{r=1}^{n}\frac{p(a_r)}{f'(a_r)} = 0$

22. Prove that $\frac{n!}{(x+1)\cdots(x+n)} \equiv \frac{C_1^n}{x+1} - \frac{2C_2^n}{x+2} + \frac{3C_3^n}{x+3} - \cdots + (-1)^{n+1}\frac{nC_n^n}{x+n}$.
Hence show that $\frac{1}{n+1} = \frac{C_1^n}{2} - \frac{2C_2^n}{3} + \frac{3C_3^n}{4} - \cdots + (-1)^{n+1}\frac{n}{1+n}$.

22. $Q(x) = (x + 1)(x + 2) \ldots (x + n)$; $Q'(x) = \sum_{r=1}^{n}\prod_{\substack{k=1\\ k\neq r}}^{n}(x+k)$

$$Q'(-r) = \prod_{\substack{k=1\\ k\neq r}}^{n}(-r+k) = (-r + 1)(-r + 2) \cdots (-3)(-2)(-1)1\cdot 2\cdot\cdots\cdot(n - r) = (-1)^{r-1}(r - 1)!(n - r)!$$

$$\frac{n!}{(x+1)\cdots(x+n)} = \sum_{r=1}^{n}\frac{n!}{Q'(-r)(x+r)} = \sum_{r=1}^{n}\frac{n!}{(-1)^{r-1}(r-1)!(n-r)!(x+r)}$$

$$= \sum_{r=1}^{n}\frac{n!}{r!(n-r)!}\cdot\frac{(-1)^{r+1}r}{(x+r)} = \sum_{r=1}^{n}\frac{(-1)^{r+1}r\cdot C_r^n}{(x+r)}$$

$$= \frac{C_1^n}{x+1} - \frac{2C_2^n}{x+2} + \frac{3C_3^n}{x+3} - \cdots + (-1)^{n+1}\frac{nC_n^n}{x+n}$$

Put $x = 1 \Rightarrow \frac{1}{n+1} = \frac{C_1^n}{2} - \frac{2C_2^n}{3} + \frac{3C_3^n}{4} - \cdots + (-1)^{n+1}\frac{n}{1+n}$

23. Aids to Advanced Level Pure Mathematics Part 2 p.74 Q27 (b)

Assume the identity $\frac{n!}{x(x+1)\cdots(x+n)}=\sum_{r=0}^{n}\frac{(-1)^r c_r}{x+r}$. Deduce the following:

(a) $\frac{c_0}{1}-\frac{c_1}{2}+\frac{c_2}{3}-\cdots+\frac{(-1)^n c_n}{n+1}=\frac{1}{n+1}$.

(b) $\frac{c_0}{2}-\frac{c_1}{3}+\frac{c_2}{4}-\cdots+\frac{(-1)^n c_n}{n+2}=\frac{1}{(n+1)(n+2)}$.

(c) $\frac{c_0}{1\cdot 2}-\frac{c_1}{2\cdot 3}+\frac{c_2}{3\cdot 4}-\cdots+\frac{(-1)^n c_n}{(n+1)(n+2)}=\frac{1}{n+2}$.

(d) $\frac{c_0}{2\cdot 3}-\frac{c_1}{3\cdot 4}+\frac{c_2}{4\cdot 5}-\cdots+\frac{(-1)^n c_n}{(n+2)(n+3)}=\frac{1}{(n+2)(n+3)}$.

(e) $\frac{c_0}{1\cdot 2\cdot 3}-\frac{c_1}{2\cdot 3\cdot 4}+\frac{c_2}{3\cdot 4\cdot 5}-\cdots+\frac{(-1)^n c_n}{(n+1)(n+2)(n+3)}=\frac{1}{2(n+3)}$.

23. (a) Put $x = 1$ in $\frac{n!}{x(x+1)\cdots(x+n)}=\sum_{r=0}^{n}\frac{(-1)^r c_r}{x+r}$

$$\frac{n!}{(n+1)!}=\sum_{r=0}^{n}\frac{(-1)^r c_r}{1+r}$$

$$\frac{c_0}{1}-\frac{c_1}{2}+\frac{c_2}{3}-\cdots+\frac{(-1)^n c_n}{n+1}=\frac{1}{n+1} \quad \cdots\cdots (1)$$

(b) Put $x = 2 \Rightarrow \frac{c_0}{2}-\frac{c_1}{3}+\frac{c_2}{4}-\cdots+\frac{(-1)^n c_n}{n+2}=\frac{1}{(n+1)(n+2)} \quad \cdots\cdots (2)$

(c) (1) – (2): $\frac{c_0}{1\cdot 2}-\frac{c_1}{2\cdot 3}+\frac{c_2}{3\cdot 4}-\cdots+\frac{(-1)^n c_n}{(n+1)(n+2)}=\frac{1}{n+2}$

(d) Put $x = 3 \Rightarrow \frac{c_0}{3}-\frac{c_1}{4}+\frac{c_2}{5}-\cdots+\frac{(-1)^n c_n}{n+3}=\frac{2}{(n+1)(n+2)(n+3)} \quad \cdots\cdots (3)$

(2) – (3): $\frac{c_0}{2\cdot 3}-\frac{c_1}{3\cdot 4}+\frac{c_2}{4\cdot 5}-\cdots+\frac{(-1)^n c_n}{(n+2)(n+3)}=\frac{1}{(n+2)(n+3)}$

(e) (c) – (d): $\frac{2c_0}{1\cdot 2\cdot 3}-\frac{2c_1}{2\cdot 3\cdot 4}+\frac{2c_2}{3\cdot 4\cdot 5}-\cdots+\frac{(-1)^n 2c_n}{(n+1)(n+2)(n+3)}=\frac{1}{n+3}$

$$\frac{c_0}{1\cdot 2\cdot 3}-\frac{c_1}{2\cdot 3\cdot 4}+\frac{c_2}{3\cdot 4\cdot 5}-\cdots+\frac{(-1)^n c_n}{(n+1)(n+2)(n+3)}=\frac{1}{2(n+3)}$$

24. Aids to Advanced Level Pure Mathematics Part 2 p.55 Example 2

(a) Prove that $\dfrac{(x-1)(x-2)\cdots(x-n)}{(x+1)(x+2)\cdots(x+n)}=1+\sum_{r=1}^{n}\dfrac{(-1)^{n-r+1}(n+r)!}{(n-r)!r!(r-1)!(x+r)}$.

(b) Deduce that $\sum_{r=1}^{n}\dfrac{(-1)^{r+1}(n+r)!}{(r!)^2(n-r)!}=\begin{cases}0 & \text{when } n \text{ is even,}\\ 2 & \text{when } n \text{ is odd.}\end{cases}$

24. (a) As the degree of the numerator and the denominator are the same, we may divide the numerator by the denominator to obtain:

$$\frac{(x-1)(x-2)\cdots(x-n)}{(x+1)(x+2)\cdots(x+n)}=1+\frac{(x-1)(x-2)\cdots(x-n)-(x+1)(x+2)\cdots(x+n)}{(x+1)(x+2)\cdots(x+n)}$$

Let it be $1+\dfrac{A_1}{x+1}+\dfrac{A_2}{x+2}+\cdots+\dfrac{A_n}{x+n}=1+\sum_{r=1}^{n}\dfrac{A_r}{x+r}$, where A_r are constants for $1 \le r \le n$.

$$(x-1)(x-2)\ldots(x-n)-(x+1)(x+2)\ldots(x+n) \equiv \sum_{r=1}^{n}A_r\prod_{\substack{s=1\\ s\ne r}}^{n}(x+s)$$

Put $x=-r$, $(-r-1)(-r-2)\cdots(-r-n) \equiv A_r(-r+1)(-r+2)\cdots(-1)(1)(2)\cdots(-r+n)$

$$A_r=\frac{(-1)^n(r+1)(r+2)\cdots(r+n)}{(-1)^{r-1}(r-1)!(n-r)!}=\frac{(-1)^{n-r+1}(n+r)!}{(r-1)!r!(n-r)!}$$

Hence $\dfrac{(x-1)(x-2)\cdots(x-n)}{(x+1)(x+2)\cdots(x+n)}=1+\sum_{r=1}^{n}\dfrac{(-1)^{n-r+1}(n+r)!}{(n-r)!r!(r-1)!(x+r)}$

(b) Put $x=0$ in the above identity:

$$\frac{(-1)^n n!}{n!}=1+\sum_{r=1}^{n}\frac{(-1)^{n-r+1}(n+r)!}{(n-r)!r!(r-1)!r}=1+(-1)^n\sum_{r=1}^{n}\frac{(-1)^{-r+1}(n+r)!}{(n-r)!r!r!}$$

$$\sum_{r=1}^{n}\frac{(-1)^{r+1}(n+r)!}{(r!)^2(n-r)!}=1-(-1)^n=\begin{cases}0 & \text{when } n \text{ is even,}\\ 2 & \text{when } n \text{ is odd.}\end{cases}$$

25. Aids to Advanced Level Pure Mathematics Part 2 p.69 Q6(*b*)

Prove that $\dfrac{1}{(x^2+1)(x^7-1)}=\dfrac{x-1}{2(x^2+1)}+\dfrac{1}{14(x-1)}+\dfrac{1}{7}\sum_{k=1}^{3}\dfrac{x\sec\frac{2k\pi}{7}-1}{x^2-2x\cos\frac{2k\pi}{7}+1}$.

Deduce that $\sec\dfrac{2\pi}{7}+\sec\dfrac{4\pi}{7}+\sec\dfrac{6\pi}{7}=-4$ by letting $x\to\infty$.

26. Aids to Advanced Level Pure Mathematics Part 2 p.70 Q11

Let $P(x)$ be a polynomial of degree n. Prove that

$P(x+y)=\dfrac{y(y+1)\cdots(y+n)}{n!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\dfrac{P(x-r)}{y+r}$, where $\binom{n}{r}=\dfrac{n!}{r!(n-r)!}$ and $\binom{n}{0}=1$.

26. Consider $\dfrac{P(x+y)}{y(y+1)\cdots(y+n)}$, regard it as a rational function in y.

Degree of numerator = n , while degree of denominator = $n+1$

By partial fraction theorem, $\dfrac{P(x+y)}{y(y+1)\cdots(y+n)}=\sum_{r=0}^{n}\dfrac{P(x-r)}{\frac{\mathrm{d}}{\mathrm{d}y}\left[y(y+1)\cdots(y+n)\right]\Big|_{y=-r}}\cdot\dfrac{1}{y+r}$

$$\frac{P(x+y)}{y(y+1)\cdots(y+n)}=\sum_{r=0}^{n}\frac{P(x-r)}{y+r}\cdot\prod_{\substack{k=0\\k\neq r}}^{n}\frac{1}{-r+k}=\sum_{r=0}^{n}\frac{1}{(-1)^r\,r!(n-r)!}\cdot\frac{P(x-r)}{y+r}$$

$$\frac{P(x+y)}{y(y+1)\cdots(y+n)}=\sum_{r=0}^{n}\frac{(-1)^r}{n!}\cdot\frac{n!}{r!(n-r)!}\cdot\frac{P(x-r)}{y+r}=\sum_{r=0}^{n}\frac{(-1)^r}{n!}\cdot\binom{n}{r}\cdot\frac{P(x-r)}{y+r}$$

$$\therefore\ P(x+y)=\frac{y(y+1)\cdots(y+n)}{n!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{P(x-r)}{y+r}$$

27. Aids to Advanced Level Pure Mathematics Part 2 p.77 Q36 (vi)

Evaluate $\dfrac{a}{b}+\dfrac{a(a+x)}{b(b+x)}+\dfrac{a(a+x)(a+2x)}{b(b+x)(b+2x)}+\cdots+\dfrac{a(a+x)(a+2x)\cdots[a+(n-1)x]}{b(b+x)(b+2x)\cdots[b+(n-1)x]}$, where $x\neq b-a$

27. Let $u_r=\dfrac{a(a+x)(a+2x)\cdots[a+(r-1)x]}{b(b+x)(b+2x)\cdots[b+(r-1)x]}$, $v_r=\dfrac{a(a+x)(a+2x)\cdots[a+(r-1)x]}{b(b+x)(b+2x)\cdots[b+(r-1)x]}(a+rx)$

$$v_r-v_{r-1}=\frac{a(a+x)(a+2x)\cdots[a+(r-1)x](a+rx)}{b(b+x)(b+2x)\cdots[b+(r-1)x]}-\frac{a(a+x)(a+2x)\cdots[a+(r-2)x][a+(r-1)x]}{b(b+x)(b+2x)\cdots[b+(r-2)x]}$$

$$=\frac{a(a+x)(a+2x)\cdots[a+(r-1)x]}{b(b+x)(b+2x)\cdots[b+(r-1)x]}\cdot\{(a+rx)-[b+(r-1)x]\}=u_r\,(a-b+x),\text{ for } r\geq 2$$

$$\therefore\ \sum_{r=2}^{n}(v_r-v_{r-1})=(a+b-x)\sum_{r=2}^{n}u_r$$

$$\therefore\ \sum_{r=1}^{n}u_r=\frac{1}{a+b-x}\sum_{r=2}^{n}(v_r-v_{r-1})+u_1=\frac{1}{a+b-x}(v_n-v_1)+u_1$$

$$=\frac{1}{a+b-x}\left\{\frac{a(a+x)\cdots(a+nx)}{b(b+x)\cdots[b+(n-1)x]}-\frac{a(a+x)}{b}\right\}+\frac{a}{b}$$

$$=\frac{1}{a+b-x}\left\{\frac{a(a+x)\cdots(a+nx)}{b(b+x)\cdots[b+(n-1)x]}-\frac{a(a+x)}{b}+\frac{a(a+b-x)}{b}\right\}$$

$$=\frac{1}{a+b-x}\left\{\frac{a(a+x)\cdots(a+nx)}{b(b+x)\cdots[b+(n-1)x]}-a\right\}$$

28. Aids to Advanced Level Pure Mathematics Part 2 p.71 Q14

(a) Show that when n is a positive integer

$$\frac{(1+x)^n}{(1-x)^4} \equiv \frac{2^n}{(1-x)^4} - \frac{n \cdot 2^{n-1}}{(1-x)^3} + \frac{\frac{n(n-1)}{2!} \cdot 2^{n-2}}{(1-x)^2} - \frac{\frac{n(n-1)(n-2)}{3!} \cdot 2^{n-3}}{1-x} + \phi(x),$$

where $\phi(x)$ is a polynomial in x of degree $n - 4$.

(b) For a positive integer n, let $\frac{(1+x)^n}{(1-x)^3} \equiv a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. By using (a) or otherwise, prove that $a_0 + a_1 + a_2 + \cdots + a_{n-1} = \frac{n}{3}(n+2)(n+7)2^{n-4}$.

29. Aids to Advanced Level Pure Mathematics Part 2 p.56 Example 3

(a) Show that $\frac{1}{(1-x)(1-x^2)(1-x^3)} = \frac{17}{72(1-x)} + \frac{1}{4(1-x)^2} + \frac{1}{6(1-x)^3} + \frac{1}{8(1+x)} + \frac{1}{9(1-\omega x)} + \frac{1}{9(1-\omega^2 x)}$,

where $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$.

(b) Let $N(n)$ denote, for any given integer n, the number of solutions of the equation $x + 2y + 3z = n$ in non-negative integer x, y, z (e.g. $N(n) = 0$ for $n < 0$, $N(0) = 1$, $N(1) = 1$, $N(2) = 2, \cdots$, etc.).

Show that $N(n)$ is the coefficient of t^n in the expansion of $\frac{1}{(1-t)(1-t^2)(1-t^3)}$.

(c) By considering the coefficient of t^n in the expansion of $\frac{1-t^6}{(1-x)(1-x^2)(1-x^3)}$ in ascending powers of t , or otherwise, prove that $N(n) - N(n - 6) = n$, where $n > 0$.

(d) By using (a), prove that $N(n) = \frac{(n+3)^2}{12} - \frac{7}{72} + \frac{(-1)^n}{8} + \frac{2}{9}\cos\frac{2n\pi}{3}$

29. (a) $\because 1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$

$\therefore 1 - x^3 = (1 - x)(1 + x + x^2) = (1 - x)(1 - \omega x)(1 - \omega^2 x)$

Let $\frac{1}{(1-x)(1-x^2)(1-x^3)} = \frac{A_1}{(1-x)^3} + \frac{A_2}{(1-x)^2} + \frac{A_3}{1-x} + \frac{A_4}{1+x} + \frac{A_5}{1-\omega x} + \frac{A_6}{1-\omega^2 x}$

$\therefore 1 \equiv A_1(1 + x)(1 - \omega x)(1 - \omega^2 x) + A_2(1- x)(1 + x)(1 - \omega x)(1 - \omega^2 x) +$
$A_3(1 - x)^2(1 + x)(1 - \omega x)(1 - \omega^2 x) + A_4(1 - x)^3(1 - \omega x)(1 - \omega^2 x) +$
$A_5(1 - x)^3(1 + x)(1 - \omega^2 x) + A_6(1 - x)^3(1 + x)(1 - \omega x)$

Put $x = -1$, $A_4 = \frac{1}{8}$

Put $x = \omega^2$, $1 = A_5(1 - \omega^2)^3(1 + \omega^2)(1 - \omega^4) = A_5(1 - 3\omega^2 + 3\omega^4 - \omega^6)(-\omega)(1 - \omega)$

$\Rightarrow 1 = A_5(1 - 3\omega^2 + 3\omega - 1)(-\omega + \omega^2) = A_5(- 3\omega^2 + 3\omega)(-\omega + \omega^2)$

$\Rightarrow 1 = 3\omega^2 A_5(-\omega + 1)(-1 + \omega) = -3\omega^2 A_5(1 - \omega)(1 - \omega) = -3\omega^2 A_5(1 - 2\omega + \omega^2)$

$\Rightarrow 1 = -3\omega^2 A_5(-\omega - 2\omega) = 9\omega^2 A_5(\omega) = 9A_5 \Rightarrow A_5 = \frac{1}{9}$

Put $x = \omega$, $1 = A_6(1 - \omega)^3(1 + \omega)(1 - \omega^2) = A_6(1 - \omega)^4(1 + \omega)^2 = A_6(1 - \omega)^4(-\omega^2)^2$

$\Rightarrow 1 = A_6(1 - 4\omega + 6\omega^2 - 4\omega^3 + \omega^4)(\omega) = \omega A_6(1 - 4\omega + 6\omega^2 - 4 + \omega)$

$\Rightarrow 1 = \omega A_6(-3 - 3\omega + 6\omega^2) = -3A_6(\omega + \omega^2 - 2) = -3A_6(-1 - 2) = 9A_6 \Rightarrow A_6 = \frac{1}{9}$

Let $y = 1 - x$, then $x = 1 - y$

$$\frac{1}{(1-x)(1-x^2)(1-x^3)} = \frac{1}{(1-x)(1-x)(1+x)(1-x)(1+x+x^2)} = \frac{1}{y^3(2-y)(2-y+1-2y+y^2)}$$

$$=\frac{1}{y^3(2-y)(3-3y+y^2)}=\frac{A_1}{y^3}+\frac{A_2}{y^2}+\frac{A_3}{y}+\frac{A_4}{2-y}+\frac{A_5}{1-\omega(1-y)}+\frac{A_6}{1-\omega^2(1-y)}$$

Multiply by y^3: $\frac{1}{(2-y)(3-3y+y^2)}=A_1+A_2y+A_3y^2+\frac{A_4y^3}{2-y}+\frac{A_5y^3}{1-\omega(1-y)}+\frac{A_6y^3}{1-\omega^2(1-y)}$

$$\frac{1}{(2-y)(3-3y+y^2)}=\frac{1}{6}\cdot\frac{1}{\left(1-\frac{y}{2}\right)\left(1-y+\frac{1}{3}y^2\right)}$$

$$=\frac{1}{6}\cdot\left(1+\frac{y}{2}+\frac{y^2}{4}+\cdots\right)\left[1+\left(y-\tfrac{1}{3}y^2\right)+\left(y-\tfrac{1}{3}y^2\right)^2+\cdots\right]$$

$$=\frac{1}{6}\cdot\left(1+\frac{y}{2}+\frac{y^2}{4}+\cdots\right)\left(1+y+\frac{2}{3}y^2+\cdots\right)$$

$$=\frac{1}{6}\cdot\left(1+\frac{3y}{2}+\frac{17y^2}{12}+\cdots\right)$$

$A_1=\frac{1}{6}$, $A_2=\frac{1}{4}$, $A_3=\frac{17}{72}$

$$\therefore \frac{1}{(1-x)(1-x^2)(1-x^3)}=\frac{17}{72(1-x)}+\frac{1}{4(1-x)^2}+\frac{1}{6(1-x)^3}+\frac{1}{8(1+x)}+\frac{1}{9(1-\omega x)}+\frac{1}{9(1-\omega^2x)}$$

(b) Obviously, $N(n)$ is equal to the total number of homogeneous products of the form $a^xb^{2y}c^{2y}$, where $x + 2y + 3z = n$. But this total number of products is equal to the number of terms in t^n in the expansion of

$(1 + at + a^2t^2 + \cdots + a^rt^r + \cdots)(1 + b^2t + b^4t^4 + \cdots + b^rt^{2r} + \cdots)(1 + c^3t^3 + c^6t^6 + \cdots + c^{3r}t^{3r} + \cdots)$

Hence if we put $a = b = c = 1$, then each of the products $a^xb^{2y}c^{2y}$ becomes 1 and the total number of products becomes the coefficient of t^n in the expansion of

$(1 + t + t^2 + \ldots + t^r + \cdots)(1 + t^2 + t^4 + \cdots + t^{2r} + \cdots)(1 + t^3 + t^6 + \cdots + t^{3r} + \cdots)$

$$=\frac{1}{(1-t)(1-t^2)(1-t^3)}.$$

(c) Following the same argument as in (b), we see that $N(n-6)$ is the coefficient of t^n in the expansion of $t^6(1 + t + t^2 + \cdots + t^r + \cdots)(1 + t^2 + t^4 + \cdots + t^{2r} + \cdots)(1 + t^3 + t^6 + \cdots + t^{3r} + \cdots)$

$$=\frac{t^6}{(1-t)(1-t^2)(1-t^3)}$$

However, $\frac{1-t^6}{(1-t)(1-t^2)(1-t^3)}=\frac{1-t+t^2}{(1-t)^2}=(1-t+t^2)[1+2t+\cdots+(r+1)t^r+\cdots]$

Thus, the coefficient of $t^n = (n + 1) - n + (n - 1) = n$, so $N(n) - N(n - 6) = n$.

(d) By (a), $\frac{1}{(1-t)(1-t^2)(1-t^3)}=\frac{17}{72(1-t)}+\frac{1}{4(1-t)^2}+\frac{1}{6(1-t)^3}+\frac{1}{8(1+t)}+\frac{1}{9(1-\omega t)}+\frac{1}{9(1-\omega^2t)}$

Assuming convergence, we can expand each of the term in an infinite series:

$$\frac{17}{72(1-t)}=\frac{17}{72}\left(1+t+t^2+\cdots+t^n+\cdots\right)$$

$$\frac{1}{4(1-t)^2}=\frac{1}{4}\left[1+2t+3t^2+\cdots+(n+1)t^n+\cdots\right]$$

$$\frac{1}{6(1-t)^3}=\frac{1}{6}\left[1+3t+6t^2+\frac{1}{2}(n+1)(n+2)t^n+\cdots\right]$$

$$\frac{1}{8(1+t)}=\frac{1}{8}\left[1-t+t^2-\cdots+(-1)^nt^n+\cdots\right]$$

$$\frac{1}{9(1-\omega t)}=\frac{1}{9}\left(1+\omega t+\omega^2t^2+\cdots+\omega^n t^n+\cdots\right)$$

$$\frac{1}{9\left(1-\omega^2 t\right)}=\frac{1}{9}\left(1+\omega^2 t+\omega^4t^2+\cdots+\omega^{2n} t^n+\cdots\right)$$

Hence $N(n)$ = coefficient of t^n

$$=\frac{17}{72}+\frac{1}{4}(n+1)+\frac{1}{12}(n+1)(n+2)+\frac{1}{8}(-1)^n+\frac{1}{9}\left(\omega^n+\omega^{2n}\right)$$

$$=\frac{17+18(n+1)+6(n+1)(n+2)}{72}+\frac{1}{8}(-1)^n+\frac{1}{9}\left(\omega^n+\omega^{-n}\right)$$

$$=\frac{17+18n+18+6n^2+18n+12}{72}+\frac{1}{8}(-1)^n+\frac{1}{9}\left(cis\frac{2n\pi}{3}+cis-\frac{2n\pi}{3}\right)$$

$$=\frac{6n^2+36n+47}{72}+\frac{1}{8}(-1)^n+\frac{1}{9}\left(cis\frac{2n\pi}{3}+cis-\frac{2n\pi}{3}\right)$$

$$=\frac{6\left(n^2+6n+9\right)-7}{72}+\frac{1}{8}(-1)^n+\frac{1}{9}\cos\frac{2n\pi}{3}$$

$$=\frac{(n+3)^2}{12}-\frac{7}{72}+\frac{(-1)^n}{8}+\frac{2}{9}\cos\frac{2n\pi}{3}$$

30. Mastering A.L. Pure Mathematics Volume 1 p.180 Q27

(a) Express in partial fractions the function $\dfrac{1-t^2}{(1-at)(1-bt)}$, where a, b are non-zero and

(i) $a \neq b$;

(ii) $a = b$

(b) By taking $a=\dfrac{1}{b}=e^{i\theta}$ and a suitable value of t deduce that if $0 < \phi < \dfrac{\pi}{2}$, then

$$\frac{\cos\phi}{1-\sin\phi\cos\theta}=1+\sum_{n=1}^{\infty}2\tan^n\frac{\phi}{2}\cos n\theta$$

31. Mastering A.L. Pure Mathematics Volume 1 p.180 Q28

Express $\dfrac{(a-b)^2}{(1-ax)^2(1-bx)}$ in terms of partial fractions.

If the function is expanded in powers of x, find the coefficient of x^n and state the range of values of x for which the expansion is valid.

Deduce that the expression $(n + 1)a^{n+2} - (n + 2)a^{n+1} b + b^{n+2}$ contains $(a - b)^2$ as a factor, where n is a positive integer.

32. Mastering A.L. Pure Mathematics Volume 1 p.179 Q24

If $a_1, a_2, \cdots, a_n$ are distinct and if $p(x)$ is a polynomial of degree $< 2n$, show that

$$\frac{p(n)}{(x-a_1)^2\cdots(x-a_n)^2}=\sum_{r=1}^{n}\left[\frac{A_r}{x-a_r}+\frac{B_r}{(x-a_r)^2}\right],$$

where $B_r=\dfrac{p(a_r)}{\prod\limits_{\substack{j=1\\ j\neq r}}^{n}(a_r-a_j)^2}$, $A_r=\dfrac{p'(a_r)}{\prod\limits_{\substack{j=1\\ j\neq r}}^{n}(a_r-a_j)^2}-2B_r\sum\limits_{\substack{j=1\\ j\neq r}}^{n}\dfrac{1}{a_r-a_j}$.

33. Aids to Advanced Level Pure Mathematics Part 2 p.54 Example 1

(a) Consider the proper irreducible rational function $\frac{\mathrm{f}(x)}{x^n\mathrm{g}(x)}$, where n is a positive integer and g(x) is a polynomial not containing x as a factor.

Suppose $\frac{\mathrm{f}(x)}{x^n\mathrm{g}(x)} \equiv \frac{A_0}{x^n} + \frac{A_1}{x^{n-1}} + \cdots + \frac{A_{n-1}}{x} + \frac{h(x)}{g(x)}$ …… (*)

for some constants $A_0, A_1, \cdots, A_{n-1}$ and some polynomial $h(x)$.

Show that $A_0, A_1, \cdots, A_{n-1}$ are the first n coefficients in the expansion (subject to convergence) of $\frac{\mathrm{f}(x)}{\mathrm{g}(x)}$ in ascending powers of x.

(b) Using (a), resolve $\frac{1}{x(x-2)(x-1)^{2n}}$ into partial fractions when n is a positive integer.

33. (a) Multiply (*) by x^n, then $\frac{\mathrm{f}(x)}{\mathrm{g}(x)} \equiv A_0 + A_1x + \cdots + A_{n-1}x^{n-1} + \frac{x^n\mathrm{h}(x)}{\mathrm{g}(x)}$.

Now $\frac{\mathrm{h}(x)}{\mathrm{g}(x)}$ can be expanded as a series of the form $B_0 + B_1x + B_2x^2 + \cdots$

Therefore, $\frac{\mathrm{f}(x)}{\mathrm{g}(x)} \equiv A_0 + A_1x + \cdots + A_{n-1}x^{n-1} + B_0x^n + B_1x^{n+1} + B_2x^{n+2} + \cdots$.

This shows that $A_0, A_1, A_2, \cdots, A_{n-1}$ are the first n coefficients of the series.

(b) Put $x - 1 = y$, then $\frac{1}{x(x-2)(x-1)^{2n}} = \frac{1}{(y+1)(y-1)y^{2n}}$

Suppose that $\frac{1}{(y+1)(y-1)y^{2n}} \equiv \frac{A}{y+1} + \frac{B}{y-1} + \frac{A_{2n-1}}{y} + \frac{A_{2n-2}}{y^2} + \cdots + \frac{A_0}{y^{2n}}$.

$\therefore\ 1 \equiv A(y-1)y^{2n} + B(y+1)y^{2n} + (y^2-1)(A_{2n-1}y^{2n-1} + A_{2n-2}y^{2n-2} + \ldots + A_0)$

Put $y = -1$, then $A = -\frac{1}{2}$; put $y = 1$, then $B = \frac{1}{2}$

To find $A_0, A_1, \cdots, A_{2n-1}$, we observe that, by ($a$), they are the first $2n$ coefficients in the expansion of $\frac{1}{y^2-1}$.

$$\frac{1}{y^2-1} = -\frac{1}{1-y^2} = -1 - y^2 - y^4 - y^6 - \cdots - y^{2r} - \cdots - y^{2n} - \cdots$$

$\therefore\ A_0 = A_2 = A_4 = \cdots = A_{2n-2} = -1$ and $A_1 = A_3 = \cdots = A_{2n-1} = 0$

So that $\frac{1}{(y+1)(y-1)y^{2n}} \equiv -\frac{1}{2(y+1)} + \frac{1}{2(y-1)} - \frac{1}{y^2} - \frac{1}{y^4} - \cdots - \frac{1}{y^{2n}}$

$$\frac{1}{x(x-2)(x-1)^{2n}} \equiv -\frac{1}{2x} + \frac{1}{2(x-2)} - \frac{1}{(x-1)^2} - \frac{1}{(x-1)^4} - \cdots - \frac{1}{(x-1)^{2n}}$$

34. Mastering A.L. Pure Mathematics Volume 1 p.172 Example 14

Express in partial fractions $\dfrac{1}{\left(x^2-a^2\right)^n}$, where n is a positive integer.

34. In the Binomial expansion, for $|h| < 1$,

$$(1-h)^{-n} = 1+(-n)(-h)+\frac{(-n)(-n-1)}{1\times 2}(-h)^2+\frac{(-n)(-n-1)(-n-2)}{1\cdot 2\cdot 3}(-h)^3+\cdots$$

$$= 1+C_1^n h+C_2^{n+1}h^2+C_3^{n+2}h^3+\cdots+C_r^{n+r-1}h^r+\cdots$$

$\therefore$ coefficient of h^r in $(1-h)^{-n}$ is $C_r = C_r^{n+r-1}$

Let $\dfrac{1}{\left(x^2-a^2\right)^n}=\dfrac{A_0}{(x-a)^n}+\dfrac{A_1}{(x-a)^{n-1}}+\cdots+\dfrac{A_{n-1}}{x-a}+\dfrac{B_0}{(x+a)^n}+\dfrac{B_1}{(x+a)^{n-1}}+\cdots+\dfrac{B_{n-1}}{x+a}$

Let $y=-x$, $\dfrac{1}{\left(y^2-a^2\right)^n}=\dfrac{(-1)^n A_0}{(y+a)^n}+\dfrac{(-1)^{n-1}A_1}{(y+a)^{n-1}}+\cdots-\dfrac{A_{n-1}}{y+a}+\dfrac{(-1)^n B_0}{(y-a)^n}+\dfrac{(-1)^{n-1}B_1}{(y-a)^{n-1}}+\cdots-\dfrac{B_{n-1}}{y-a}$

$\therefore B_0=(-1)^n A_0$, $B_1=(-1)^{n-1}A_1$, $\cdots$, $B_r=(-1)^{n-r}A_r$, $\cdots$, $B_{n-1}=-A_{n-1}$

Let $t=x-a \Rightarrow x=t+a$

$$\frac{1}{t^n(t+2a)^n}=\frac{A_0}{t^n}+\frac{A_1}{t^{n-1}}+\cdots+\frac{A_{n-1}}{t}+\frac{B_0}{(t+2a)^n}+\frac{B_1}{(t+2a)^{n-1}}+\cdots+\frac{B_{n-1}}{t+2a}$$

$$\frac{1}{(t+2a)^n}=A_0+A_1t+\cdots+A_{n-1}t^{n-1}+t^n\left[\frac{B_0}{(t+2a)^n}+\frac{B_1}{(t+2a)^{n-1}}+\cdots+\frac{B_{n-1}}{t+2a}\right]$$

$\dfrac{1}{(2a)^n}\cdot\dfrac{1}{\left(1+\frac{t}{2a}\right)^n}=A_0+A_1t+\cdots+A_{n-1}t^{n-1}$ + terms involving powers of t^n, t^{n+1}, $\cdots$

$\dfrac{1}{(2a)^n}\left[1+C_1\left(-\dfrac{t}{2a}\right)+C_2\left(-\dfrac{t}{2a}\right)^2+\cdots+C_r\left(-\dfrac{t}{2a}\right)^r+\cdots+C_{n-1}\left(-\dfrac{t}{2a}\right)^{n-1}+\cdots\right]=A_0+A_1t+\ldots+A_{n-1}t^{n-1}$+terms involving powers $\geq n+1$

$\therefore A_0=\dfrac{1}{(2a)^n}$, $A_1=-\dfrac{C_1}{(2a)^{n+1}}$, $\cdots$, $A_r=\dfrac{(-1)^r C_r}{(2a)^{n+r}}$, $\cdots$, $A_{n-1}=\dfrac{(-1)^{n-1}C_{n-1}}{(2a)^{2n-1}}$.

$B_0=\dfrac{(-1)^n}{(2a)^n}$, $B_1=\dfrac{(-1)^n C_1}{(2a)^{n+1}}$, $\cdots$, $B_r=\dfrac{(-1)^n C_r}{(2a)^{n+r}}$, $\cdots$, $B_{n-1}=\dfrac{(-1)^n C_{n-1}}{(2a)^{2n-1}}$.

$$\frac{1}{\left(x^2-a^2\right)^n}=\sum_{r=0}^{n-1}\left[\frac{A_r}{(x-a)^{n-r}}+\frac{B_r}{(x+a)^{n-r}}\right]=\sum_{r=0}^{n-1}\left[\frac{(-1)^r C_r}{(2a)^{n+r}(x-a)^{n-r}}+\frac{(-1)^n C_r}{(2a)^{n+r}(x+a)^{n-r}}\right]$$

$$\frac{1}{\left(x^2-a^2\right)^n}=\frac{(-1)^n}{(2a)^{2n}}\sum_{r=0}^{n-1}C_r^{n+r-1}\left[\left(\frac{-2a}{x-a}\right)^{n-r}+\left(\frac{2a}{x+a}\right)^{n-r}\right]$$

35. Aids to Advanced Level Pure Mathematics Part 2 p.70 Q12

If $\alpha \neq \beta$, show that $\dfrac{1}{(x-\alpha)^n(x-\beta)^n} \equiv \dfrac{(-1)^n}{(\alpha-\beta)^{2n}} \sum_{r=0}^{n-1} C_r \left\{ \left(\dfrac{\beta-\alpha}{x-\alpha}\right)^{n-r} + \left(\dfrac{\alpha-\beta}{x-\beta}\right)^{n-r} \right\}$

where C_r = coefficient of h^r in $(1-h)^{-n}$.

35. Let $\dfrac{1}{(x-\alpha)^n(x-\beta)^n} = \dfrac{A_0}{(x-\alpha)^n} + \dfrac{A_1}{(x-\alpha)^{n-1}} + \cdots + \dfrac{A_{n-1}}{x-\alpha} + \dfrac{B_0}{(x-\beta)^n} + \dfrac{B_1}{(x-\beta)^{n-1}} + \cdots + \dfrac{B_{n-1}}{x-\beta}$

Let $y = \alpha+\beta-x$, LHS $= \dfrac{1}{(\alpha+\beta-y-\alpha)^n(\alpha+\beta-y-\beta)^n} = \dfrac{1}{(\beta-y)^n(\alpha-y)^n} = \dfrac{1}{(y-\alpha)^n(y-\beta)^n}$

RHS $= \dfrac{A_0}{(\beta-y)^n} + \dfrac{A_1}{(\beta-y)^{n-1}} + \cdots + \dfrac{A_{n-1}}{\beta-y} + \dfrac{B_0}{(\alpha-y)^n} + \dfrac{B_1}{(\alpha-ya)^{n-1}} + \cdots + \dfrac{B_{n-1}}{\alpha-y}$

$= \dfrac{(-1)^n A_0}{(y-\beta)^n} + \dfrac{(-1)^{n-1} A_1}{(y-\beta)^{n-1}} + \cdots - \dfrac{A_{n-1}}{y-\beta} + \dfrac{(-1)^n B_0}{(y-\alpha)^n} + \dfrac{(-1)^{n-1} B_1}{(y-\alpha)^{n-1}} + \cdots - \dfrac{B_{n-1}}{y-\alpha}$

$\therefore B_0 = (-1)^n A_0$, $B_1 = (-1)^{n-1} A_1$, $\cdots$, $B_r = (-1)^{n-r} A_r$, $\cdots$, $B_{n-1} = -A_{n-1}$

Let $t = x - \alpha \Rightarrow x = t + \alpha$

$$\frac{1}{t^n(t+\alpha-\beta)^n} = \frac{A_0}{t^n} + \frac{A_1}{t^{n-1}} + \cdots + \frac{A_{n-1}}{t} + \frac{B_0}{(t+\alpha-\beta)^n} + \frac{B_1}{(t+\alpha-\beta)^{n-1}} + \cdots + \frac{B_{n-1}}{t+\alpha-\beta}$$

$$\frac{1}{(t+\alpha-\beta)^n} = A_0 + A_1 t + \cdots + A_{n-1}t^{n-1} + t^n\left[\frac{B_0}{(t+\alpha-\beta)^n} + \frac{B_1}{(t+\alpha-\beta)^{n-1}} + \cdots + \frac{B_{n-1}}{t+\alpha-\beta}\right]$$

$\dfrac{1}{(\alpha-\beta)^n} \cdot \dfrac{1}{\left(1+\frac{t}{\alpha-\beta}\right)^n} = A_0 + A_1 t + \cdots + A_{n-1}t^{n-1}$ + terms involving powers of t^n, t^{n+1}, $\cdots$

$$\frac{1}{(\alpha-\beta)^n}\left[1 + C_1\left(-\frac{t}{\alpha-\beta}\right) + C_2\left(-\frac{t}{\alpha-\beta}\right)^2 + \cdots + C_r\left(-\frac{t}{\alpha-\beta}\right)^r + \cdots + C_{n-1}\left(-\frac{t}{\alpha-\beta}\right)^{n-1} + \cdots\right]$$

$= A_0 + A_1 t + \cdots + A_{n-1}t^{n-1}$ + terms of powers $\geq n+1$

$\therefore A_0 = \dfrac{1}{(\alpha-\beta)^n}$, $A_1 = -\dfrac{C_1}{(\alpha-\beta)^{n+1}}$, $\cdots$, $A_r = \dfrac{(-1)^r C_r}{(\alpha-\beta)^{n+r}}$, $\cdots$, $A_{n-1} = \dfrac{(-1)^{n-1} C_{n-1}}{(\alpha-\beta)^{2n-1}}$,

$B_0 = \dfrac{(-1)^n}{(\alpha-\beta)^n}$, $B_1 = \dfrac{(-1)^n C_1}{(\alpha-\beta)^{n+1}}$, $\cdots$, $B_r = \dfrac{(-1)^n C_r}{(\alpha-\beta)^{n+r}}$, $\cdots$, $B_{n-1} = \dfrac{(-1)^n C_{n-1}}{(\alpha-\beta)^{2n-1}}$.

$$\frac{1}{(x-\alpha)^n(x-\beta)^n} = \sum_{r=0}^{n-1}\left[\frac{A_r}{(x-\alpha)^{n-r}} + \frac{B_r}{(x-\beta)^{n-r}}\right] = \sum_{r=0}^{n-1}\left[\frac{(-1)^r C_r}{(\alpha-\beta)^{n+r}(x-\alpha)^{n-r}} + \frac{(-1)^n C_r}{(\alpha-\beta)^{n+r}(x-\beta)^{n-r}}\right]$$

$$\frac{1}{(x-\alpha)^n(x-\beta)^n} = \frac{(-1)^n}{(\alpha-\beta)^{2n}}\sum_{r=0}^{n-1} C_r^{n+r-1}\left[\left(\frac{\beta-\alpha}{x-\alpha}\right)^{n-r} + \left(\frac{\alpha-\beta}{x-\beta}\right)^{n-r}\right]$$

1956 Paper 1 Q4 (a)

Resolve $\dfrac{x^2}{x^4-2x^2+1}$ into partial fractions.

$$\frac{x^2}{x^4-2x^2+1}\equiv\frac{x^2}{\left(x^2-1\right)^2}\equiv\left(\frac{x}{x^2-1}\right)^2\equiv\left[\frac{x}{(x+1)(x-1)}\right]^2\equiv\left[\frac{1}{2(x+1)}+\frac{1}{2(x-1)}\right]^2$$

$$\frac{x^2}{x^4-2x^2+1}\equiv\frac{1}{4(x+1)^2}+\frac{1}{2(x-1)(x+1)}+\frac{1}{4(x-1)^2}$$

$$\frac{x^2}{x^4-2x^2+1}\equiv\frac{1}{4(x+1)^2}+\frac{1}{4}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)+\frac{1}{4(x-1)^2}\equiv\frac{1}{4(x+1)^2}+\frac{1}{4(x-1)}-\frac{1}{4(x+1)}+\frac{1}{4(x-1)^2}$$

1957 Paper 1 Q5 (*b*)

Obtain the first four terms in the expansion of the following function in ascending powers of x:

$\dfrac{1}{1-3x-4x^2}$.

$$\frac{1}{1-3x-4x^2}=\frac{1}{(1+x)(1-4x)}=\frac{A}{1+x}+\frac{B}{1-4x}$$

$1\equiv A(1-4x)+B(1+x)$

Put $x=-1\Rightarrow A=\dfrac{1}{5}$

Put $x=\dfrac{1}{4}\Rightarrow B=\dfrac{4}{5}$

$$\therefore\ \frac{1}{1-3x-4x^2}=\frac{1}{5(1+x)}+\frac{4}{5(1-4x)}=\frac{1}{5}\left(1-x+x^2-x^3+\cdots\right)+\frac{4}{5}\left(1+4x+16x^2+64x^3+\cdots\right)$$

$$=1+3x+13x^2+51x^3+\cdots$$

1959 Paper 1 Q5(b)

Find a polynomial $g(x)$ in x which satisfies the equation $\dfrac{P(x)}{Q(x)}=\dfrac{1}{x+2}+\dfrac{g(x)}{\left(x^2+1\right)^4}$, where

$P(x)=x^8+x^7+6x^6+3x^5+12x^4+4x^2-7x-13$, $Q(x)=(x+2)(x^2+1)^4$.

Hence resolve $\dfrac{P(x)}{Q(x)}$ into partial fractions.

After taking common denominators of R.H.S. and compare the numerators on both sides,

$P(x)\equiv(x^2+1)^4+(x+2)g(x)$

$$g(x)=\frac{P(x)-\left(x^2+1\right)^4}{(x+2)}=x^6+3x^4+7$$

	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x	1
	1	1	6	3	12	0	4	−7	−13
−)	1		4		6		4		1
(−2)		1	2	3	6	0	0	−7	−14
	+)		−2	0	−6	0	0	0	14
		1	0	3	0	0	0	−7	0
		x^6	x^5	x^4	x^3	x^2	x	1	

$$\frac{P(x)}{Q(x)}\equiv\frac{1}{x+2}+\frac{t^3+3t^2-7}{(t+1)^4},\ \text{where } t=x^2$$

$$=\frac{1}{x+2}+\frac{A}{t+1}+\frac{B}{(t+1)^2}+\frac{C}{(t+1)^3}+\frac{D}{(t+1)^4}$$

For $\dfrac{t^3+3t^2-7}{(t+1)^4}\equiv\dfrac{A}{t+1}+\dfrac{B}{(t+1)^2}+\dfrac{C}{(t+1)^3}+\dfrac{D}{(t+1)^4}$

$t^3+3t^2-7\equiv A(t+1)^3+B(t+1)^2+C(t+1)+D$

Put $t=-1\Rightarrow D=-5$

Differentiate once and put $t=-1\Rightarrow(3t^2+6t)|_{t=-1}=C\Rightarrow C=-3$

Differentiate once and put $t=-1\Rightarrow(6t+6)|_{t=-1}=2B\Rightarrow B=0$

Compare coefficient of t^3: $A=1$

$$\therefore\ \frac{P(x)}{Q(x)}\equiv\frac{1}{x+2}+\frac{1}{x^2+1}-\frac{3}{\left(x^2+1\right)^3}-\frac{5}{\left(x^2+1\right)^4}$$

1961 Paper 1 Q5(a)

Find the values of the constants A, B, C and D so that $\dfrac{3+x^2}{(1-x)^2(1+x^2)} \equiv \dfrac{A}{1-x} + \dfrac{B}{(1-x)^2} + \dfrac{C+Dx}{1+x^2}$.

$$\frac{3+x^2}{(1-x)^2(1+x^2)} \equiv \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C+Dx}{1+x^2}$$

$3 + x^2 \equiv A(1 - x)(1 + x^2) + B(1 + x^2) + (C + Dx)(1 - x)^2$

Put $x = 1 \Rightarrow B = 2$

Put $x = i \Rightarrow 2 = (C - Di)(1 - i)^2 \Rightarrow 2 = (C - Di)(1 - 2i - 1) \Rightarrow 1 = D - Ci \Rightarrow C = 0, D = 1$

Compare coefficient of x^3: $-A + D = 0 \Rightarrow A = 1$

$$\therefore \quad \frac{3+x^2}{(1-x)^2(1+x^2)} \equiv \frac{1}{1-x} + \frac{2}{(1-x)^2} + \frac{x}{1+x^2}$$

1962 Paper 1 Q3(a)

Resolve the expression $\dfrac{x^6 - x^2 + 1}{(x-1)^3}$ into partial fractions.

	x^6	x^5	x^4	x^3	x^2	x	1
	1	0	0	0	–1	0	1
3		3	9	18	30		
–3			–3	–9	–18	–30	
1				1	3	6	10
	1	3	6	10	14	–24	11
	x^3	x^2	x	1	x^2	x	1

$$\frac{x^6 - x^2 + 1}{(x-1)^3} \equiv x^3 + 3x^2 + 6x + 10 + \frac{14x^2 - 24x + 11}{(x-1)^3}$$

Suppose $\dfrac{14x^2 - 24x + 11}{(x-1)^3} \equiv \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x-1)^3}$

Take the common denominators of the R.H.S. and compare the numerators of both sides

$14x^2 - 24x + 11 \equiv A(x - 1)^2 + B(x - 1) + C$

Put $x = 1 \Rightarrow C = 1$

Differentiate once and put $x = 1 \Rightarrow B = 28 - 24 = 4$

Compare coefficient of x^2: $a = 14$

$$\therefore \quad \frac{x^6 - x^2 + 1}{(x-1)^3} \equiv x^3 + 3x^2 + 6x + 10 + \frac{14}{x-1} + \frac{4}{(x-1)^2} + \frac{1}{(x-1)^3}$$

1964 Paper 1 Q2(b)

Resolve $\frac{x}{(x+1)(x+2)(x+3)}$ into partial fractions. Hence find the sum $\sum_{n=1}^{\infty}\frac{n}{(n+1)(n+2)(n+3)}$.

$\frac{x}{(x+1)(x+2)(x+3)} \equiv \sum_{r=1}^{3}\frac{-r}{Q'(-r)(x+r)}$, where $Q(x) = (x + 1)(x + 2)(x + 3)$

$$= \frac{-1}{(-1+2)(-1+3)(x+1)} - \frac{2}{(-2+1)(-2+3)(x+2)} - \frac{3}{(-3+1)(-3+2)(x+3)}$$

$$= -\frac{1}{2(x+1)} + \frac{2}{x+2} - \frac{3}{2(x+3)}$$

Let $S_N = \sum_{n=1}^{N}\frac{n}{(n+1)(n+2)(n+3)} = \sum_{n=1}^{N}\left[-\frac{1}{2(n+1)} + \frac{2}{n+2} - \frac{3}{2(n+3)}\right]$

$$= \sum_{n=1}^{N} -\frac{1}{2(n+1)} + \sum_{n=1}^{N}\frac{2}{n+2} - \sum_{n=1}^{N}\frac{3}{2(n+3)}$$

$$= \sum_{n=2}^{N+1} -\frac{1}{2n} + \sum_{n=3}^{N+2}\frac{2}{n} - \sum_{n=4}^{N+3}\frac{3}{2n}$$

$$= -\frac{1}{4} - \frac{1}{6} + \frac{2}{3} + \frac{2}{N+2} - \frac{3}{2(N+2)} - \frac{3}{2(N+3)} + \sum_{n=4}^{N+1}\left(-\frac{1}{2n} + \frac{2}{n} - \frac{3}{2n}\right)$$

$$= \frac{1}{4} + \frac{2}{N+2} - \frac{3}{2(N+2)} - \frac{3}{2(N+3)} + \sum_{n=4}^{N+1} 0$$

$$\sum_{n=1}^{\infty}\frac{n}{(n+1)(n+2)(n+3)} = \lim_{N\to\infty}\sum_{n=1}^{N}\frac{n}{(n+1)(n+2)(n+3)} = \lim_{N\to\infty}\left[\frac{1}{4} + \frac{2}{N+2} - \frac{3}{2(N+2)} - \frac{3}{2(N+3)}\right] = \frac{1}{4}$$

1966 Paper 1 Q7(a)

Let $g(x)$ be a quadratic polynomial and a, b, c distinct constants.

If $\frac{g(x)}{(x-a)(x-b)(x-c)} \equiv \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$, where A, B, C are constants, express A in terms of a, b, c and $g(a)$. Hence or otherwise resolve $\frac{x^2}{(x-1)(x-2)(x-3)}$ into partial fractions.

$A = \frac{g(a)}{(a-b)(a-c)}$

$$\frac{x^2}{(x-1)(x-2)(x-3)} \equiv \frac{1}{(1-2)(1-3)(x-1)} + \frac{4}{(2-1)(2-3)(x-2)} + \frac{9}{(3-1)(3-2)(x-3)}$$

$$\equiv \frac{1}{2(x-1)} - \frac{4}{x-2} + \frac{9}{2(x-3)}$$

1967 Paper 1 Q4

Let n be any positive integer, and a_r, b_r, the coefficients of x^r in $(1 + x)^n$ and $(1 + x)^{n+2}$ respectively. Prove that

(a) $b_{r+2} = a_r + 2a_{r+1} + a_{r+2}$ if $0 \le r \le n - 2$,

(b) $$\frac{n!}{x(x+1)\cdots(x+n)} = \sum_{r=0}^{n} \frac{(-1)^r a_r}{x+r},$$

(c) $$\frac{a_0}{x(x+1)(x+2)} - \frac{a_1}{(x+1)(x+2)(x+3)} + \cdots + \frac{(-1)^n a_n}{(x+n)(x+n+1)(x+n+2)} = \frac{(n+2)!}{2x(x+1)\cdots(x+n+2)},$$

(a) $(1 + x)^{n+2} = (1 + 2x + x^2)(1 + x)^n$

$$\sum_{r=0}^{n+2} b_r x^r = \left(1+2x+x^2\right)\sum_{r=0}^{n} a_r x^r$$

Compare coefficients of x^{r+2}, $0 \le r \le n - 2$

$b_{r+2} = a_r + 2a_{r+1} + a_{r+2}$

(b) $$\frac{n!}{x(x+1)\cdots(x+n)} = \sum_{r=0}^{n} \frac{n!}{Q'(-r)(x+r)}$$, where $Q(x) = x(x + 1) \cdots (x + n)$

$$= \sum_{r=0}^{n} \frac{n!}{(x+r)} \prod_{\substack{s=0 \\ s\neq r}}^{n} \frac{1}{-r+s}$$

$$= \sum_{r=0}^{n} \frac{n!}{(-r)(-r+1)\cdots(-1)(1)(2)\cdots(n-r)(x+r)}$$

$$= \sum_{r=0}^{n} \frac{n!}{(-1)^r r!(n-r)!(x+r)}$$

$$= \sum_{r=0}^{n} \frac{(-1)^r C_r^n}{x+r} = \sum_{r=0}^{n} \frac{(-1)^r a_r}{x+r}$$

(c) Note that $a_0 = C_0^n = 1 = C_0^{n+2}$

$a_1 + 2a_0 = C_1^n + 2\,C_0^n = n + 2 = C_1^{n+2} = b_1$

$a_{n-1} + 2a_n = C_{n-1}^n + 2\,C_n^n = n + 2 = C_{n+1}^{n+2} = b_{n+1}$

$a_n = C_n^n = 1 = C_{n+2}^{n+2} = b_{n+2}$

$$\text{LHS} = \frac{a_0}{x(x+1)(x+2)} - \frac{a_1}{(x+1)(x+2)(x+3)} + \cdots + \frac{(-1)^n a_n}{(x+n)(x+n+1)(x+n+2)}$$

$$= \sum_{r=0}^{n} \frac{(-1)^r a_r}{(x+r)(x+r+1)(x+r+2)}$$

$$= \sum_{r=0}^{n} \left[\frac{(-1)^r a_r}{2(x+r)} - \frac{(-1)^r a_r}{x+r+1} + \frac{(-1)^r a_r}{2(x+r+2)}\right]$$ (By partial faction theorem)

$$=\sum_{r=-2}^{n-2}\frac{(-1)^{r+2}a_{r+2}}{2(x+r+2)}-\sum_{r=-1}^{n-1}\frac{(-1)^{r+1}a_{r+1}}{x+r+2}+\sum_{r=0}^{n}\frac{(-1)^{r}a_{r}}{2(x+r+2)}$$

$$=\frac{a_0}{2x}-\frac{a_1}{2(x+1)}-\frac{a_0}{x+1}+\sum_{r=0}^{n-2}(-1)^r\frac{a_{r+2}+2a_{r+1}+a_{r+2}}{2(x+r+2)}-\frac{(-1)^n a_n}{x+n+1}+\frac{(-1)^{n-1}a_{n-1}}{2(x+n+1)}+\frac{(-1)^n a_n}{2(x+n+2)}$$

$$=\frac{b_0}{2x}-\frac{b_1}{2(x+1)}+\sum_{r=2}^{n}(-1)^r\frac{b_r}{2(x+r+2)}+\frac{(-1)^{n+1}b_{n+1}}{2(x+n+1)}+\frac{(-1)^{n+2}b_n}{2(x+n+2)}$$

$$=\sum_{r=0}^{n+2}\frac{(-1)^r b_r}{2(x+r+2)}$$

$$=\frac{(n+2)!}{2x(x+1)\cdots(x+n+2)}$$

1970 Paper 2 Q8 (a)

Find the values of A, B so that $\frac{-2x+4}{(x^2+1)(x-1)^2}\equiv\frac{Ax+B}{x^2+1}-\frac{2}{x-1}+\frac{1}{(x-1)^2}$ for all x.

Hence or otherwise find the indefinite integral $\int\frac{-2x+4}{(x^2+1)(x-1)^2}\,\mathrm{d}x$.

After taking common denominators of R.H.S. and compare the numerators on both sides,

$-2x + 4 \equiv (Ax + B)(x - 1)^2 - 2(x^2 + 1)(x - 1) + x^2 + 1$

$(Ax + B)(x - 1)^2 \equiv -2x + 4 + 2x^3 - 2x^2 + 2x - 2 - x^2 - 1$

$\equiv 2x^3 - 3x^2 + 1$

	x^3	x^2	x	1
	2	−3	0	1
2		4	2	
−1			−2	−1
	2	1	0	0
	x	1	x	1

$Ax + B \equiv \frac{2x^3-3x^2+1}{(x-1)^2} \equiv 2x + 1$

$\therefore A = 2, B = 1$

$$\int\frac{-2x+4}{(x^2+1)(x-1)^2}\,\mathrm{d}x=\int\frac{2x+1}{x^2+1}\,\mathrm{d}x-\int\frac{2}{x-1}\,\mathrm{d}x+\int\frac{1}{(x-1)^2}\,\mathrm{d}x$$

$= \ln(x^2 + 1) + \tan^{-1} x - 2\ln|x - 1| - \frac{1}{x-1} + C$, where C is a constant.

1973 Paper 1 Q4

(a) Let $A(x)$ be a polynomial of degree n in x, with real coefficients and n real roots $x_1, x_2, \cdots, x_n$.

Prove that $\sum_{r=1}^{n}\frac{1}{x-x_r}=\frac{A'(x)}{A(x)}$, where $A'(x)$ is the derivative of $A(x)$. Hence or otherwise, prove that $\sum_{r=1}^{n}\frac{1}{(x-x_r)^2}=\frac{A'(x)^2-A(x)A''(x)}{A(x)^2}$.

(b) Resolve $\frac{2x-1}{(x-1)^2}$ into partial fractions.

(c) Let x_1, x_2, x_3, x_4 be the roots of the polynomial $B(x) = x^4 - 10x + 1$.
(You can assume that all the roots of $B(x)$ are real.) Using (a) and (b) or otherwise, evaluate the sum $\sum_{r=1}^{n}\frac{2x_r-1}{(x_r-1)^2}$.

(a) Induction on n. $n = 1$, the result is obvious.

Suppose $\frac{\frac{d}{dx}(x-x_1)\cdots(x-x_k)}{(x-x_1)\cdots(x-x_k)}=\sum_{r=1}^{k}\frac{1}{x-x_r}$.

When $n = k + 1$, let $P(x)=\frac{d}{dx}\left[(x-x_1)\cdots(x-x_k)(x-x_{k+1})\right]=(x-x_{k+1})A'(x)+A(x)$

Let $Q(x) = (x - x_1)\cdots(x - x_k)(x - x_{k+1})$

$$\frac{P(x)}{Q(x)}\equiv\frac{(x-x_{k+1})A'(x)+A(x)}{A(x)(x-x_{k+1})}\equiv\frac{A'(x)}{A(x)}+\frac{1}{x-x_{k+1}}\equiv\sum_{r=1}^{k+1}\frac{1}{x-x_r}$$

$\therefore$ It is also true for $n = k + 1$. By mathematical induction, it is true for all positive integer n.

To prove that $\sum_{r=1}^{n}\frac{1}{(x-x_r)^2}=\frac{A'(x)^2-A(x)A''(x)}{A(x)^2}$.

Differentiate the given identity once and multiply by –1 gives the required result.

(b) Let $\frac{2x-1}{(x-1)^2}\equiv\frac{A}{x-1}+\frac{B}{(x-1)^2}$

After taking common denominators of RHS and compare the numerators on both sides,

$2x - 1 \equiv A(x - 1) + B$

$A = 2, -1 = -A + B \Rightarrow B = 1$

$\therefore\ \frac{2x-1}{(x-1)^2}\equiv\frac{2}{x-1}+\frac{1}{(x-1)^2}$

(c) $\sum_{r=1}^{n}\frac{2x_r-1}{(x_r-1)^2}=2\sum_{r=1}^{n}\frac{1}{x_r-1}+\sum_{r=1}^{n}\frac{1}{(x_r-1)^2}$ (by the result of (b))

$$=-2\left.\frac{A'(x)}{A(x)}\right|_{x=1}+\left.\frac{A'(x)^2-A(x)A''(x)}{A(x)^2}\right|_{x=1}$$

$$=-2\left.\frac{4x^3-20x}{x^4-10x^2+1}\right|_{x=1}+\left.\frac{(4x^3-20x)^2-(x^4-10x^2+1)(12x^2-20)}{(x^4-10x^2+1)^2}\right|_{x=1}$$

$$=\frac{(-2)(-16)}{-8}+\frac{(-16)^2-(-8)(-8)}{(-8)^2}$$

$= -4 + 4 - 1 = -1$

1975 Paper 1 Q2

(a) Let $b_1, \cdots, b_n$ be real numbers, $B(x) = (x - b_1)(x - b_2) \cdots (x - b_n)$ and $B'(x)$ the derivative of $B(x)$. Show that $b_1, \cdots, b_n$ are all distinct if and only if $B'(b_1), \cdots, B'(b_n)$ are all non-zero.

(b) Now suppose that $b_1, \cdots, b_n$ are all distinct. For a polynomial $A(x)$ of degree < n in x, use induction to prove that $\dfrac{A(x)}{B(x)} = \sum_{r=1}^{n} \dfrac{A(b_r)}{B'(b_r)(x - b_r)}$.

(c) Let $1 \le p \le n - 1$, where p is an integer. By using (b) or otherwise, resolve $\dfrac{x^p}{(x+1)(x+2)\cdots(x+n)}$ into partial fractions.

Hence show that $\dfrac{1^p}{1!(n-1)!} - \dfrac{2^p}{2!(n-2)!} + \cdots + \dfrac{(-1)^{n-2}(n-1)^p}{(n-1)!1!} + \dfrac{(-1)^{n-1}n^p}{n!} = 0$.

Find the value of $\dfrac{1^n}{1!(n-1)!} - \dfrac{2^n}{2!(n-2)!} + \cdots + \dfrac{(-1)^{n-2}(n-1)^n}{(n-1)!1!} + \dfrac{(-1)^{n-1}n^n}{n!}$.

(a) (⇒) $B'(x) = \sum_{r=1}^{n} \prod_{\substack{k=1 \\ k \ne r}}^{n} (x - b_k)$

$\because b_1, \cdots, b_n$ are all distinct

$\therefore b_i - b_k \ne 0$ for $i \ne k$

$B'(b_i) = \prod_{\substack{k=1 \\ k \ne i}}^{n} (b_i - b_k) \ne 0$ for all i: $1 \le i \le n$

(⇐) Suppose one of $B'(b_1), \cdots, B'(b_n)$ is zero

say, $B'(b_1) = 0$

$\prod_{k=2}^{n} (b_1 - b_k) = 0$

$\Rightarrow b_k = b_1$ for some $k \ne 1$

contradicting the fact that $b_1, \cdots, b_n$ are distinct.

(b) Induction on degree of $B(x) = n$

$n = 1$, $A(x) = a$, which is a constant with degree = 0

$\dfrac{A(x)}{B(x)} = \dfrac{a}{x - b_1} = \sum_{r=1}^{1} \dfrac{A(b_r)}{B'(b_r)(x - b_r)}$, the result is obvious.

Suppose it is true for $n = k$

When $n = k + 1$, $A(x)$ = polynomial of degree $\le k$

$B(x) = (x - b_1) \cdots (x - b_k)(x - b_{k+1})$

Assume the <u>existence</u> of partial fraction,

$\dfrac{A(x)}{B(x)} \equiv \dfrac{p}{x - b_{k+1}} + \dfrac{\mathrm{f}(x)}{\mathrm{g}(x)}$, where $\mathrm{g}(x) = (x - b_1) \cdots (x - b_k)$, $\mathrm{f}(x)$ is a polynomial

After taking common denominators of R.H.S. and compare the numerators on both sides,

$A(x) \equiv pg(x) + (x - b_{k+1})\,\mathrm{f}(x) \cdots\cdots (1)$

Compare the degrees of both sides

$\deg A(x) \le k \Rightarrow \deg (x - b_{k+1})\,\mathrm{f}(x) \le k$

$\therefore \deg \mathrm{f}(x) \le k - 1$

Put $x = b_{k+1}$ into both sides of (1) $\Rightarrow p = \dfrac{A(b_{k+1})}{g(b_{k+1})} = \dfrac{A(b_{k+1})}{\prod_{r=1}^{k}(b_{k+1}-b_r)} = \dfrac{A(b_{k+1})}{B'(b_{k+1})}$

By induction assumption, $\dfrac{\mathrm{f}(x)}{\mathrm{g}(x)} \equiv \sum_{r=1}^{k} \dfrac{\mathrm{f}(b_r)}{\mathrm{g}'(b_r)(x-b_r)}$

Now put $x = b_r$ into (1), for $1 \le r \le k$

$A(b_r) = p\mathrm{g}(b_r) + \mathrm{f}(b_r)(b_r - b_{k+1})$

$A(b_r) = \qquad \mathrm{f}(b_r)(b_r - b_{k+1})$

$$\frac{A(b_r)}{B'(b_r)} = \frac{\mathrm{f}(b_r)(b_r-b_{k+1})}{\prod_{\substack{i=1\\ i\ne r}}^{k+1}(b_r-b_i)} = \frac{\mathrm{f}(b_r)}{\prod_{\substack{i=1\\ i\ne r}}^{k}(b_r-b_i)} = \frac{\mathrm{f}(b_r)}{\mathrm{g}'(b_r)}$$

$$\therefore \quad \frac{A(x)}{B(x)} \equiv \frac{A(b_{k+1})}{B'(b_{k+1})}\cdot\frac{1}{x-b_{k+1}} + \sum_{r=1}^{k}\frac{A(b_r)}{B'(b_r)(x-b_r)} \equiv \sum_{r=1}^{k+1}\frac{A(b_r)}{B'(b_r)(x-b_r)}$$

$\therefore$ It is also true for $n = k + 1$, by M.I., it is true for all $n \ge 1$.

(*c*) $$\frac{x^p}{(x+1)(x+2)\cdots(x+n)}$$

$\equiv \sum_{r=1}^{n}\dfrac{A(-r)}{B'(-r)(x+r)}$, where $A(x) = x^p$, $B(x) = (x + 1)(x + 2) \cdots (x + n)$

$$\equiv \sum_{r=1}^{n}\frac{(-r)^p}{(x+r)}\cdot\frac{1}{\prod_{\substack{s=1\\ s\ne r}}^{n}(-r+s)} \equiv \sum_{r=1}^{n}\frac{(-r)^p}{(x+r)}\cdot\frac{1}{(-r+1)(-r+2)\cdots(-1)(1)(2)\cdots(n-r)}$$

$$\equiv \sum_{r=1}^{n}\frac{(-r)^p}{(x+r)}\cdot\frac{1}{(-1)^{r-1}(r-1)!(n-r)!} \equiv \sum_{r=1}^{n}\frac{(-1)^{p-r+1}r^p}{(r-1)!(n-r)!(x+r)}$$

Put $x = 0$ into both sides,

$$0 = (-1)^p\left[\frac{1^p}{1!(n-1)!} - \frac{2^p}{2!(n-2)!} + \cdots + \frac{(-1)^{n-2}(n-1)^p}{(n-1)!1!} + \frac{(-1)^{n-1}n^p}{n!}\right]$$

Hence result.

To find the value of $\dfrac{1^n}{1!(n-1)!} - \dfrac{2^n}{2!(n-2)!} + \cdots + \dfrac{(-1)^{n-2}(n-1)^n}{(n-1)!1!} + \dfrac{(-1)^{n-1}n^n}{n!}$.

Consider $\dfrac{x^n}{(x+1)(x+2)\cdots(x+n)} \equiv 1 + \dfrac{x^n-(x+1)(x+2)\cdots(x+n)}{(x+1)(x+2)\cdots(x+n)}$.

$$\equiv 1 + \sum_{r=1}^{n}\frac{A(-r)}{B'(-r)(x+r)} \equiv 1 + \sum_{r=1}^{n}\frac{(-1)^{n-r+1}r^n}{(r-1)!(n-r)!(x+r)}$$

Put $x = 0 \Rightarrow 0 = 1 + (-1)^n\left[\dfrac{1^n}{1!(n-1)!} - \dfrac{2^n}{2!(n-2)!} + \cdots + \dfrac{(-1)^{n-2}(n-1)^n}{(n-1)!1!} + \dfrac{(-1)^{n-1}n^n}{n!}\right]$

$$\therefore \quad \frac{1^n}{1!(n-1)!} - \frac{2^n}{2!(n-2)!} + \cdots + \frac{(-1)^{n-2}(n-1)^n}{(n-1)!1!} + \frac{(-1)^{n-1}n^n}{n!} = (-1)^{n-1}$$

1976 Paper 1 Q7

Let n be a positive integer, and a_k and b_k the coefficients of x^k in $(1 + x)^n$ and $(1 + x)^{n+2}$ respectively.

(a) Show that, for $0 \le k \le n - 2$, $b_{k+2} = a_k + 2a_{k+1} + a_{k+2}$.

(b) Show that $\dfrac{n!}{x(x+1)\cdots(x+n)} = \sum_{k=0}^{n} \dfrac{(-1)^k a_k}{x+k}$.

(c) Using (a) and (b) or otherwise, show that $\sum_{k=0}^{n} \dfrac{(-1)^k a_k}{(x+k)(x+k+1)(x+k+2)} = \dfrac{(n+2)!}{2x(x+1)\cdots(x+n+2)}$.

This question is identical to 1967 Paper 1 Q4

Modified from 1979 Paper 1 Q3

Let $a_1, a_2, \cdots, a_n$ be n (≥ 2) distinct real numbers, $f(x) = (x - a_1)(x - a_2) \cdots (x - a_n)$ and $f'(x)$ the derivative of $f(x)$.

(a) Express $f'(a_i)$ ($i = 1, 2, \cdots, n$) in terms of $a_1, a_2, \cdots, a_n$.

(b) Let $g(x)$ be a real polynomial of degree less than n .

(i) Show that there exist unique real numbers $A_1, A_2, \cdots, A_n$ such that

$$g(x) = \sum_{i=1}^{n} A_i \prod_{\substack{r=1 \\ r\ne i}}^{n} (x - a_r) \quad \cdots (*)$$

(ii) Using (i), or otherwise, show that if $g(x)$ is of degree less than $n - 1$, then $\sum_{i=1}^{n} \dfrac{g(a_i)}{f'(a_i)} = 0$.

(iii) By taking $a_i = i$ ($i = 1, 2, \cdots, n$) and suitable $g(x)$ in (ii), show that, for any non-negative integer $m \le n - 2$, $\sum_{i=1}^{n} (-1)^{n-i} \dfrac{i^n}{(i-1)!(n-i)!} = 0$. (Given that $0! = 1$.)

(c) If $b_1, b_2, \cdots, b_n$ are n real numbers, find a polynomial $h(x)$ of degree less than n in the form of the right hand side of (*) so that $h(a_i) = b_i$ ($i = 1, \cdots, n$) .

(a) $f'(a_i) = \prod_{\substack{k=1 \\ k\ne i}}^{n} (a_i - a_k)$

(b) (i) <u>Uniqueness</u>

By partial fraction theorem, $\dfrac{g(x)}{f(x)} = \sum_{i=n}^{n} \dfrac{g(a_i)}{f'(a_i)(x - a_i)}$.

This theorem has been proved in 1975 Paper 1 Q2(b)

After taking common denominators of R.H.S. and compare the numerators on both sides,

$g(x) = \sum_{i=1}^{n} A_i \prod_{\substack{r=1 \\ r\ne i}}^{n} (x - a_r)$, where $A_i = \dfrac{g(a_i)}{f'(a_i)}$

<u>Existence</u>

Induction on n (degree of $f(x)$)

$n = 2$, $f(x) = (x - a_1)(x - a_2)$

$\deg g(x) = 0, 1$ (or $-\infty$)

$\because$ $x - a_1$ and $x - a_2$ has no common factors

By <u>Euclidean algorithm</u>, there exist polynomials $h_1(x)$, $h_2(x)$ such that

$1 \equiv h_1(x)(x - a_1) + h_2(x)(x - a_2)$

$g(x) \equiv h_1(x)g(x)(x - a_1) + h_2(x)g(x)(x - a_2)$

By division algorithm, when $h_1(x)\ g(x) \div (x - a_2)$

$h_1(x)\ g(x) = Q(x)(x - a_2) + A_1$, where A_1 is a real constant

$\therefore\ g(x) \equiv [Q(x)(x - a_2) + A_1](x - a_1) + h_2(x)g(x)(x - a_2)$

$g(x) \equiv A_1(x - a_1) + A_2(x - a_2)$ $\cdots\cdots$ (*), where $A_2 \equiv Q(x)(x - a_1) + h_2(x)g(x)$

By the formula (*), degree $g(x) \le 1$

in RHS of (*), deg $A_1(x - a_2) \le 1$ (As A_1 is a constant)

$\therefore$ deg $A_2(x - a_2) \le 1$

$\therefore$ deg $A_2 \le 0$

$\therefore$ A_2 is a constant

We have proved that there exist real constants A_1 and A_2 such that

$g(x) \equiv A_1(x - a_1) + A_2(x - a_2)$

$\therefore$ It is true for $n = 2$

Suppose it is true for $n = k$

When $n = k + 1$, $f(x) = (x - a_1)\cdots(x - a_k)(x - a_{k+1})$, deg $g(x) \le k$

$\because$ $(x - a_1) \cdots (x - a_k)$ and $(x - a_{k+1})$ are relatively prime

$\exists$ polynomials $h_1(x)$, $h_2(x)$ such that

$1 \equiv h_1(x)(x - a_1) \cdots (x - a_k) + h_2(x)(x - a_{k+1})$

$g(x) \equiv h_1(x)g(x)(x - a_1) \cdots (x - a_k) + h_2(x)g(x)(x - a_{k+1})$

By division algorithm, when $h_1(x)\ g(x) \div (x - a_{k+1})$

$h_1(x)\ g(x) = Q(x)(x - a_{k+1}) + A_{k+1}$, where A_{k+1} is a real constant

$\therefore\ g(x) \equiv [Q(x)(x - a_{k+1}) + A_{k+1}](x - a_1) \cdots (x - a_k) + h_2(x)g(x)(x - a_{k+1})$

$g(x) \equiv A_{k+1}(x - a_1) \cdots (x - a_k) + A(x - a_{k+1})$ $\cdots\cdots$ (**)

where $A \equiv Q(x)(x - a_1) \cdots (x - a_k) + h_2(x)g(x)$

By the formula (**), degree L.H.S. $\le k$

In R.H.S. of (**), deg $A_{k+1}(x - a_1) \cdots (x - a_k) = k$ (As A_1 is a constant)

$\therefore$ deg $A(x - a_{k+1}) \le k$

$\therefore$ deg $A_2 \le k - 1$

By induction assumption on A ,

$$A \equiv \sum_{i=1}^{k} A_i \prod_{\substack{r=1 \\ r\ne i}}^{k} (x - a_r)$$

$$\therefore\ g(x) \equiv A_{k+1}(x - a_1) \cdots (x - a_k) + (x - a_{k+1}) \sum_{i=1}^{k} A_i \prod_{\substack{r=1 \\ r\ne i}}^{k} (x - a_r)$$

$$g(x) \equiv \sum_{i=1}^{k+1} A_i \prod_{\substack{r=1 \\ r\ne i}}^{k+1} (x - a_r),\ \text{where } A_{k+1} = A$$

$\therefore$ It is also true for $n = k + 1$. By M.I., it is true for all positive integer > 1 .

(b) (ii) By (b)(i), $\frac{\mathrm{g}(a_i)}{\mathrm{f}'(a_i)} = A_i$ (uniqueness part)

$\therefore \sum_{i=1}^{n} \frac{\mathrm{g}(a_i)}{\mathrm{f}'(a_i)} = \sum_{i=1}^{n} A_i$ = coefficient of x^{n-1} in g(x) = 0 (given that deg g(x) < n – 1)

(iii) Let g(x) = x^m, where m is an integer ≤ n – 2

f(x) = (x – 1) ⋯ (x – n)

f'(i) = $(-1)^{n-i}(i-1)!(n-i)!$

$\therefore \sum_{i=1}^{n} (-1)^{n-i} \frac{i^n}{(i-1)!(n-i)!} = \sum_{i=1}^{n} \frac{\mathrm{g}(i)}{\mathrm{f}'(i)} = 0$ by (b)(ii)

(c) Let $h(x) = \sum_{i=1}^{n} b_i \prod_{\substack{k=1 \\ k\neq i}}^{n} \frac{x-a_k}{(a_i-a_k)}$, then $h(a_i) = b_i$ for i = 1, 2, ⋯ , n

1981 Paper 2 Q4

(a) Resolve $\frac{1}{(1+x)(1+2x)\cdots(1+nx)}$ into partial fractions.

(b) Use the result in (a) to prove the identity $\sum_{k=0}^{n} (-1)^{n-k} C_k^n k^n = n!$,

where C_r^n are binomial coefficients.

(c) Prove that the n^{th} derivative with respect to t of $(e^t - 1)^n$ takes the value $n!$ when t is zero.

(a) $$\frac{1}{(1+x)(1+2x)\ldots(1+nx)} = \frac{1}{n!}\cdot\frac{1}{(x+1)\left(x+\frac{1}{2}\right)\cdots\left(x+\frac{1}{n}\right)}$$

$$= \frac{1}{n!}\sum_{k=1}^{n}\frac{1}{Q'\left(-\frac{1}{k}\right)\left(x+\frac{1}{k}\right)}, \text{ where } Q(x) = (x+1)\left(x+\frac{1}{2}\right)\cdots\left(x+\frac{1}{n}\right)$$

$$= \frac{1}{n!}\sum_{k=1}^{n}\frac{k}{(1+kx)}\cdot\frac{1}{\prod_{\substack{r=1\\r\neq k}}^{n}\left(-\frac{1}{k}+\frac{1}{r}\right)} = \frac{1}{n!}\sum_{k=1}^{n}\frac{k}{(1+kx)}\cdot\prod_{\substack{r=1\\r\neq k}}^{n}\left(\frac{kr}{k-r}\right)$$

$$= \frac{1}{n!}\sum_{k=1}^{n}\frac{k}{(1+kx)}\cdot\frac{\frac{n!}{k}k^{n-1}}{(k-1)!(-1)^{n-k}(n-k)!}$$

$$= \sum_{k=1}^{n}\frac{(-1)^{n-k}k^n}{n!}\cdot\frac{n!}{k!(n-k)!}\cdot\frac{1}{(1+kx)} = \sum_{k=1}^{n}\frac{(-1)^{n-k}k^n C_k^n}{n!}\cdot\frac{1}{(1+kx)}$$

(b) Put x = 0 into both sides: $1 = \sum_{k=1}^{n}\frac{(-1)^{n-k}k^n C_k^n}{n!}$

$$\sum_{k=1}^{n}(-1)^{n-k}C_k^n k^n = n! \Rightarrow \sum_{k=0}^{n}(-1)^{n-k}C_k^n k^n = n!$$

(c) $(e^t-1)^n = \sum_{k=0}^{n}(-1)^{n-k}C_k^n e^{kt}$

Differentiate n times: $\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(e^t-1\right)^n = \sum_{k=0}^{n}(-1)^{n-k}C_k^n k^n e^{kt}$

Put t = 0, $\left.\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(e^t-1\right)^n\right|_{t=0} = \sum_{k=0}^{n}(-1)^{n-k}C_k^n k^n = n!$ by (b)

1985 Paper 1 Q3

(a) Let $a_1, a_2, \cdots, a_n$ be distinct real numbers. Suppose f(x) is a polynomial of degree less than $n - 1$ and the expression $\dfrac{\mathrm{f}(x)}{(x+a_1)(x+a_2)\cdots(x+a_n)}$ is resolved into partial fractions as $\dfrac{c_1}{x+a_1}+\dfrac{c_2}{x+a_2}+\cdots+\dfrac{c_n}{x+a_n}$, show that $c_1 + c_2 + \cdots + c_n = 0$.

(b) Let $F(x) = \dfrac{px+q}{(x+a)(x+a+1)(x+a+2)}$ be resolved into partial fractions as $\dfrac{b_1}{x+a}+\dfrac{b_2}{x+a+1}+\dfrac{b_3}{x+a+2}$. Show that for $N > 3$, $\sum_{k=1}^{N} F(k)=\dfrac{b_1}{1+a}+\dfrac{b_1+b_2}{2+a}+\dfrac{b_2+b_3}{N+a+1}+\dfrac{b_3}{N+a+2}$.

(c) Using (b), or otherwise, evaluate $\lim_{N\to\infty}\sum_{k=1}^{N}\dfrac{1}{(2k+1)(2k+3)(2k+5)}$.

(a) $\dfrac{\mathrm{f}(x)}{(x+a_1)(x+a_2)\cdots(x+a_n)}=\dfrac{c_1}{x+a_1}+\dfrac{c_2}{x+a_2}+\cdots+\dfrac{c_n}{x+a_n}$

Combining the partial fractions of the R.H.S., the numerator is

$c_1(x + a_2)(x + a_2)\cdots(x + a_n) + c_2(x + a_1)(x + a_3)\cdots(x + a_n) +\cdots+ c_n(x + a_1)(x + a_2) \cdots (x + a_{n-1})$

$= (c_1 + c_2 + \cdots + c_n)x^{n-1}$ + (terms of degree $< n - 1$)

Since f(x) is a polynomial of degree $< n - 1$

$\therefore c_1 + c_2 + \cdots + c_n = 0$

(b) $F(x)=\dfrac{px+q}{(x+a)(x+a+1)(x+a+2)}\equiv\dfrac{b_1}{x+a}+\dfrac{b_2}{x+a+1}+\dfrac{b_3}{x+a+2}$.

Compare the numerators of both sides.

$px + q \equiv b_1(x + a + 1)(x + a + 2) + b_2(x + a)(x + a + 2) + b_3(x + a)(x + a + 1)$

Compare the coefficients of x^2: $b_1 + b_2 + b_3 = 0$

$$\sum_{k=1}^{N}F(k)=\sum_{k=1}^{N}\left(\frac{b_1}{k+a}+\frac{b_2}{k+a+1}+\frac{b_3}{k+a+2}\right)$$

$$=\sum_{k=1}^{N}\frac{b_1}{k+a}+\sum_{k=1}^{N}\frac{b_2}{k+a+1}+\sum_{k=1}^{N}\frac{b_3}{k+a+2}$$

$$=\sum_{k=1}^{N}\frac{b_1}{k+a}+\sum_{k=2}^{N+1}\frac{b_2}{k+a}+\sum_{k=3}^{N+2}\frac{b_3}{k+a} \quad \text{1M}$$

$$=\frac{b_1}{1+a}+\frac{b_1}{2+a}+\frac{b_2}{2+a}+\sum_{k=2}^{N}\frac{b_1+b_2+b_3}{k+a}+\frac{b_2}{N+a+1}+\frac{b_3}{N+a+1}+\frac{b_3}{N+a+2}$$

$$=\frac{b_1}{1+a}+\frac{b_1+b_2}{2+a}+\frac{b_2+b_3}{N+a+1}+\frac{b_3}{N+a+2}$$

(c) $F(k)=\dfrac{1}{(2k+1)(2k+3)(2k+5)}=\dfrac{0k+\frac{1}{8}}{(k+\frac{1}{2})(k+\frac{3}{2})(k+\frac{5}{2})}=\dfrac{b_1}{k+\frac{1}{2}}+\dfrac{b_2}{k+\frac{3}{2}}+\dfrac{b_3}{k+\frac{5}{2}}$, $a=\dfrac{1}{2}$

Compare the numerators of both sides.

$$\frac{1}{8}=b_1\left(k+\frac{3}{2}\right)\left(k+\frac{5}{2}\right)+b_2\left(k+\frac{1}{2}\right)\left(k+\frac{5}{2}\right)+b_3\left(k+\frac{1}{2}\right)\left(k+\frac{3}{2}\right)$$

Put $k=-\dfrac{1}{2}\Rightarrow b_1=\dfrac{1}{16}$

Put $k=-\dfrac{3}{2}\Rightarrow b_2=-\dfrac{1}{8}$

Put $k=-\frac{5}{2}\Rightarrow b_3=\frac{1}{16}$

$$\sum_{k=1}^{N}F(k)=\frac{b_1}{1+a}+\frac{b_1+b_2}{2+a}+\frac{b_2+b_3}{N+a+1}+\frac{b_3}{N+a+2}=\frac{1}{24}-\frac{1}{40}-\frac{1}{8(2N+3)}+\frac{1}{8(2N+5)}$$

$$\lim_{N\to\infty}\sum_{k=1}^{N}\frac{1}{(2k+1)(2k+3)(2k+5)}=\frac{1}{24}-\frac{1}{40}-0+0=\frac{1}{60}$$

1990 Paper 1 Q2

(a) Resolve $\frac{1}{x(x+1)(x+2)}$ into partial fractions.

(b) Evaluate $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)}$.

(a) Let $\frac{1}{x(x+1)(x+2)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x+2}$

then $1\equiv A(x+1)(x+2)+Bx(x+2)+Cx(x+1)$

Put $x=0\Rightarrow A=\frac{1}{2}$

Put $x=-1\Rightarrow B=-1$

Put $x=-2\Rightarrow C=\frac{1}{2}$

$$\frac{1}{x(x+1)(x+2)}=\frac{1}{2x}-\frac{1}{x+1}+\frac{1}{2(x+2)}$$

(b)
$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)}=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{2x}-\frac{1}{x+1}+\frac{1}{2(x+2)}\right].$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{2x}-\frac{1}{x+1}+\frac{1}{2(x+2)}\right]$$

$$=\lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{2x}-\sum_{k=1}^{n}\frac{1}{x+1}+\sum_{k=1}^{n}\frac{1}{2(x+2)}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{2}\sum_{k=1}^{n}\frac{1}{x}-\sum_{k=2}^{n+1}\frac{1}{x}+\frac{1}{2}\sum_{k=3}^{n+2}\frac{1}{x}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{2}\left(1+\frac{1}{2}\right)-\left(\frac{1}{2}+\frac{1}{n+1}\right)+\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}\right)\right]=\frac{1}{4}$$

1991 Paper 1 Q2

Let $f(x) = \dfrac{1}{(x-1)(2-x)}$.

Express f(x) into partial fractions. Hence, or otherwise, determine a_k and b_k ($k = 0, 1, 2, \cdots$) such that

$f(x) = \sum_{k=0}^{\infty} a_k x^k$ when $|x| < 1$ and $f(x) = \sum_{k=0}^{\infty} \dfrac{a_k}{x^k}$ when $|x| > 2$.

$$\frac{1}{(x-1)(2-x)} = \frac{1}{x-1} + \frac{1}{2-x}$$

When $|x| < 1$, $f(x) = \dfrac{1}{x-1} + \dfrac{1}{2-x}$

$$= (-1)\frac{1}{1-x} + \frac{1}{2}\cdot\left(\frac{1}{1-\frac{x}{2}}\right)$$

$$= (-1)\sum_{k=0}^{\infty} x^k + \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^k$$

$$= \sum_{k=0}^{\infty}\left(\frac{1}{2^{k+1}} - 1\right)x^k$$

$\therefore\ a_k = \dfrac{1}{2^{k+1}} - 1$

When $|x| > 2$, $f(x) = \dfrac{1}{x-1} + \dfrac{1}{2-x}$

$$= \frac{1}{x}\cdot\frac{1}{1-\frac{1}{x}} - \frac{1}{x}\cdot\left(\frac{1}{1-\frac{2}{x}}\right)$$

$$= \frac{1}{x}\sum_{k=0}^{\infty}\left(\frac{1}{x}\right)^k - \frac{1}{x}\sum_{k=0}^{\infty}\left(\frac{2}{x}\right)^k$$

$$= \sum_{k=0}^{\infty}\frac{1}{x^{k+1}} - \sum_{k=0}^{\infty}\frac{2^k}{x^{k+1}}$$

$$= \sum_{k=0}^{\infty}\left(1-2^k\right)\frac{1}{x^{k+1}}$$

$$= \sum_{k=1}^{\infty}\left(1-2^{k-1}\right)\frac{1}{x^k}$$

$$= \sum_{k=1}^{\infty} b_k\left(\frac{1}{x^k}\right)$$

where $b_k = \begin{cases} 0 & k = 0 \\ 1-2^{k-1} & k = 1, 2, \cdots \end{cases}$

1991 Paper 1 Q4

Let $a_1, a_2, \cdots, a_n$ be n distinct non-zero real numbers, where $n \ge 2$.

(a) Define $P(n) = a_1 \frac{(x-a_2)\cdots(x-a_n)}{(a_1-a_2)\cdots(a_1-a_n)} + \cdots + a_i \frac{(x-a_1)\cdots(x-a_{i-1})(x-a_{i+1})\cdots(x-a_n)}{(a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_n)}$

$+ \cdots + a_n \frac{(x-a_1)\cdots(x-a_{n-1})}{(a_n-a_1)\cdots(a_1-a_{n-1})}$.

(i) Evaluate $P(a_i)$ for $i = 1, 2, \cdots, n$.

(ii) Show that the equation $P(x) - x = 0$ has n distinct roots.

(iii) Deduce that $P(x) - x = 0$ for all $x \in \mathbb{R}$.

(b) Prove that $\frac{1}{(a_1-a_2)\cdots(a_1-a_n)} + \cdots + \frac{1}{(a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_n)}$

$+ \cdots + \frac{1}{(a_n-a_1)\cdots(a_1-a_{n-1})} = 0.$

(a) (i) For $i = 1, 2, \cdots, n$,

$P(a_i) = a_i \frac{(a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_n)}{(a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_n)} = a_i$

(ii) By (a)(i), $a_1, a_2, \cdots, a_n$ are n distinct roots of $P(x) - x = 0$

(iii) Since $\deg(P(x) - x) \le n - 1$ and $P(x) - x = 0$ has n distinct roots, $\therefore P(x) - x \equiv 0$

(b) By (a) (iii), $P(0) = 0$

$\Rightarrow (a_1a_2\cdots a_n)(-1)^{n-1}\left\{\frac{1}{(a_1-a_2)\cdots(a_1-a_n)} + \cdots + \frac{1}{(a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_n)}\right.$

$\left. + \cdots + \frac{1}{(a_n-a_1)\cdots(a_1-a_{n-1})}\right\} = 0$

$\Rightarrow \frac{1}{(a_1-a_2)\cdots(a_1-a_n)} + \cdots + \frac{1}{(a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_n)} + \cdots + \frac{1}{(a_n-a_1)\cdots(a_1-a_{n-1})} = 0.$

($\because a_i \ne 0\ \forall i$)

1993 Paper 1 Q5

Express $\dfrac{x+4}{x^2+3x+2}$ in partial fractions. Hence evaluate $\displaystyle\sum_{k=2}^{\infty}\left\{\frac{1}{k-1}-\frac{k+4}{k^2+3k+2}\right\}$

Let $\dfrac{x+4}{x^2+3x+2}=\dfrac{A}{x+1}+\dfrac{B}{x+2}$

Then $x + 4 \equiv A(x + 2) + B(x + 1)$

Put $x = -1 \Rightarrow A = 3$

Put $x = -2 \Rightarrow B = -2$

$\therefore\ \dfrac{x+4}{x^2+3x+2}=\dfrac{3}{x+1}-\dfrac{2}{x+2}$

$$\sum_{k=2}^{N}\left\{\frac{1}{k-1}-\frac{k+4}{k^2+3k+2}\right\}=\sum_{k=2}^{N}\left\{\frac{1}{k-1}-\frac{3}{k+1}+\frac{2}{k+2}\right\}$$

$$=\sum_{k=2}^{N}\frac{1}{k-1}-3\sum_{k=2}^{N}\frac{1}{k+1}+2\sum_{k=2}^{N}\frac{1}{k+2}$$

$$=\sum_{k=1}^{N-1}\frac{1}{k}-3\sum_{k=3}^{N+1}\frac{1}{k}+2\sum_{k=4}^{N+2}\frac{1}{k}$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}\right)-3\left(\frac{1}{3}+\frac{1}{N}+\frac{1}{N+1}\right)+2\left(\frac{1}{N}+\frac{1}{N+1}+\frac{1}{N+2}\right)\to\ \frac{5}{6}\ \text{ as } N\to\infty$$

1995 Paper 1 Q10

Let α, β and γ be real and distinct and $(x - \alpha)(x - \beta)(x - \gamma) = x^3 + px^2 + qx + r$.

(a) Show that

(i) $\dfrac{1}{x-\alpha}+\dfrac{1}{x-\beta}+\dfrac{1}{x-\gamma}=\dfrac{3x^2+2px+q}{x^3+px^2+qx+r}$;

(ii) $3\alpha^2 + 2p\alpha + q = (\alpha - \beta)(\alpha - \gamma)$.

(b) Let f(x) be a real polynomial. Suppose $Ax^2 + Bx + C$ is the remainder when $(3x^2 + 2px + q)\text{f}(x)$ is divided by $x^3 + px^2 + qx + r$.

(i) Prove that $\dfrac{\text{f}(\alpha)}{x-\alpha}+\dfrac{\text{f}(\beta)}{x-\beta}+\dfrac{\text{f}(\gamma)}{x-\gamma}=\dfrac{Ax^2+Bx+C}{x^3+px^2+qx+r}$.

(ii) Express A, B and C in terms of α, β, γ, f(α), f(β) and f(γ).

(a) (i) $(x - \alpha)(x - \beta)(x - \gamma) = x^3 + px^2 + qx + r$ for all x.

Differentiating w.r.t. x on both sides, we have

$(x - \alpha)(x - \beta) + (x - \gamma)(x - \alpha) + (x - \beta)(x - \gamma) = 3x^2 + 2px + q \cdots (1)$

Hence $\dfrac{1}{x-\alpha}+\dfrac{1}{x-\beta}+\dfrac{1}{x-\gamma}=\dfrac{(x-\alpha)(x-\beta)+(x-\alpha)(x-\gamma)+(x-\beta)(x-\gamma)}{(x-\alpha)(x-\beta)(x-\gamma)}$

$$=\frac{3x^2+2px+q}{x^3+px^2+qx+r}$$

(ii) Put $x = \alpha$ into (1), we have $3\alpha^2 + 2p\alpha + q = (\alpha - \beta)(\alpha - \gamma)$.

(b) (i) Let $(3x^2 + 2px + q)f(x) = (x^3 + px^2 + qx + r)Q(x) + Ax^2 + Bx + C$

$= (x - \alpha)(x - \beta)(x - \gamma)Q(x) + Ax^2 + Bx + C$.

Then $(3\alpha^2 + 2p\alpha + q)f(\alpha) = A\alpha^2 + B\alpha + C \cdots\cdots (2)$

Let $\dfrac{Ax^2+Bx+C}{x^3+px^2+qx+r}=\dfrac{k_1}{x-\alpha}+\dfrac{k_2}{x-\beta}+\dfrac{k_3}{x-\gamma}$.

Then $Ax^2 + Bx + C = k_1(x - \beta)(x - \gamma) + k_2(x - \gamma)(x - \alpha) + k_3(x - \alpha)(x - \beta)$.

Put $x = \alpha$, we have $A\alpha^2 + B\alpha + C = k_1(\alpha - \beta)(\alpha - \gamma)$

By (2), $(3\alpha^2 + 2p\alpha + q)\text{f}(\alpha) = k_1(\alpha - \beta)(\alpha - \gamma)$

By (a) (ii), $k_1 = \text{f}(\alpha)$

Similarly, $k_2 = \text{f}(\beta)$ and $k_3 = \text{f}(\gamma)$.

Hence $\dfrac{\text{f}(\alpha)}{x-\alpha}+\dfrac{\text{f}(\beta)}{x-\beta}+\dfrac{\text{f}(\gamma)}{x-\gamma}=\dfrac{Ax^2+Bx+C}{x^3+px^2+qx+r}$.

(ii) From (b)(i), $Ax^2 + Bx + C = \text{f}(\alpha)(x - \beta)(x - \gamma) + \text{f}(\beta)(x - \gamma)(x - \alpha) + \text{f}(\gamma)(x - \alpha)(x - \beta)$

Equating the coefficients of x^2, x and the constant terms, we have

$A = \text{f}(\alpha) + \text{f}(\beta) + \text{f}(\gamma)$

$B = -[(\beta + \gamma)\text{f}(\alpha) + (\gamma + \alpha)\text{f}(\beta) + (\alpha + \beta)\text{f}(\gamma)]$

$C = \beta\gamma\text{f}(\alpha) + \gamma\alpha\text{f}(\beta) + \alpha\beta\text{f}(\gamma)$

2000 Paper 1 Q12

(a) Resolve $\dfrac{x^3 - x^2 - 3x + 2}{x^2(x-1)^2}$ into partial fractions.

(b) Let $P(x) = m(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)$ where $m, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ and $m \neq 0$. Prove that

(i) $\displaystyle\sum_{i=1}^{4} \frac{1}{x - \alpha_i} = \frac{P'(x)}{P(x)}$, and

(ii) $\displaystyle\sum_{i=1}^{4} \frac{1}{(x - \alpha_i)^2} = \frac{[P'(x)]^2 - P(x)P''(x)}{[P(x)]^2}$.

(c) Let $f(x) = ax^4 - bx^2 + a$ where $ab > 0$ and $b^2 > 4a^2$.

(i) Show that the four roots of $f(x) = 0$ are real and none of them is equal to 0 or 1 .

(ii) Denote the roots of $f(x) = 0$ by $\beta_1, \beta_2, \beta_3$ and β_4 .

Find $\displaystyle\sum_{i=1}^{4} \frac{\beta_i^3 - \beta_i^2 - 3\beta_i + 2}{\beta_i^2(\beta_i - 1)^2}$ in terms of a and b .

(a) Let $\dfrac{x^3 - x^2 - 3x + 2}{x^2(x-1)^2} \equiv \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1} + \dfrac{D}{(x-1)^2}$, then

$Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2 \equiv x^3 - x^2 - 3x + 2$

Put $x = 0 \Rightarrow B = 2$; put $x = 1 \Rightarrow D = -1$

Compare coefficient of x^3: $A + C = 1$

Differentiate once and put $x = 0$: $A(-1)^2 + 2B(-1) = -3$

$\Rightarrow A - 4 = -3 \Rightarrow A = 1, C = 0$

$\therefore \dfrac{x^3 - x^2 - 3x + 2}{x^2(x-1)^2} \equiv \dfrac{1}{x} + \dfrac{2}{x^2} - \dfrac{1}{(x-1)^2}$

(b) (i) $P(x) = m(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)$

$P'(x) = m[(x - \alpha_2)(x - \alpha_3)(x - \alpha_4) + (x - \alpha_1)(x - \alpha_3)(x - \alpha_4) + (x - \alpha_1)(x - \alpha_2)(x - \alpha_4)$
$+ (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)]$

$\therefore \dfrac{P'(x)}{P(x)} = \displaystyle\sum_{i=1}^{4} \frac{1}{x - \alpha_i}$

(ii) From (b)(i), $\displaystyle\sum_{i=1}^{4} \frac{1}{x - \alpha_i} = \frac{P'(x)}{P(x)}$

Differentiate both sides w.r.t. x.

$$-\sum_{i=1}^{4} \frac{1}{(x - \alpha_i)^2} = \frac{P(x)P''(x) - [P'(x)]^2}{[P(x)]^2}$$

$$\sum_{i=1}^{4} \frac{1}{(x - \alpha_i)^2} = \frac{[P'(x)]^2 - P(x)P''(x)}{[P(x)]^2}$$

(c) (i) Solve $f(x) = 0$, we have $x^2 = \dfrac{b \pm \sqrt{b^2 - 4a^2}}{2a}$

$\because b^2 > 4a^2$ and $a \neq 0$, $\therefore |b| > \sqrt{b^2 - 4a^2} > 0$

$\because ab > 0$, $\therefore$ ($a > 0$ and $b > 0$) or ($a < 0$ and $b < 0$)

If $b > 0$, then $b > \sqrt{b^2 - 4a^2} > 0$ and $a > 0$.

If $b < 0$, then $-b > \sqrt{b^2 - 4a^2} > 0$ and $a < 0$.

Both sides imply $x^2 > 0$.

Hence all roots of f(x) = 0 are real.

Besides, f(0) = $a \neq 0$ and f(1) = $2a - b \neq 0$ ($\because b^2 > 4a^2 \therefore (b+2a)(b-2a) > 0$)

$\therefore$ Both 0 and 1 are not the roots of f(x) = 0 .

(ii) $$\sum_{i=1}^{4}\frac{\beta_i^3-\beta_i^2-3\beta_i+2}{\beta_i^2(\beta_i-1)^2}=\sum_{i=1}^{4}\frac{1}{\beta_i}+2\sum_{i=1}^{4}\frac{1}{\beta_i^2}-\sum_{i=1}^{4}\frac{1}{(\beta_i-1)^2}\quad\text{(by (a))}$$

$$=-\sum_{i=1}^{4}\frac{1}{(0-\beta_i)}+2\sum_{i=1}^{4}\frac{1}{(0-\beta_i)^2}-\sum_{i=1}^{4}\frac{1}{(1-\beta_i)^2}$$

$$=-\frac{f'(0)}{f(0)}+2\frac{[f'(0)]^2-f(0)f''(0)}{[f(0)]^2}-\frac{[f'(1)]^2-f(1)f''(1)}{[f(1)]^2}\quad\text{by (b)(i)\&(ii)}$$

$\because$ f(x) = $ax^4 - bx^2 + a$, f '(x) = $4ax^3 - 2bx$ and f "(x) = $12ax^2 - 2b$

$\therefore$ f(0) = a, f '(0) = 0, f "(0) = $-2b$ and

f(1) = $2a - b$, f '(1) = $2(2a - b)$, f "(1) = $2(6a - b)$

Hence $$\sum_{i=1}^{4}\frac{\beta_i^3-\beta_i^2-3\beta_i+2}{\beta_i^2(\beta_i-1)^2}=0+2\frac{0^2-a(-2b)}{a^2}-\frac{[2(2a-b)]^2-(2a-b)2(6a-b)}{(2a-b)^2}$$

$$=\frac{4b}{a}-\frac{-4a-2b}{2a-b}$$

$$=\frac{4a^2+10ab-4b^2}{a(2a-b)}$$

2001 Paper 1 Q1

(a) Resolve $\frac{8}{x(x-2)(x+2)}$ into partial fractions.

(b) Show that $\sum_{r=3}^{2001}\frac{8}{r(r-2)(r+2)}<\frac{11}{12}$.

(a) Let $\frac{8}{x(x-2)(x+2)}=\frac{A}{x-2}+\frac{B}{x}+\frac{C}{x+2}$, then

$Ax(x + 2) + B(x - 2)(x + 2) + Cx(x - 2) = 8$

Put $x = 0 \Rightarrow B = -2$; put $x = 2 \Rightarrow A = 1$; put $x = -2 \Rightarrow C = 1$

$\therefore \frac{8}{x(x-2)(x+2)}=\frac{1}{x-2}-\frac{2}{x}+\frac{1}{x+2}$

(b) $$\sum_{r=3}^{2001}\frac{8}{r(r-2)(r+2)}=\sum_{r=3}^{2001}\left(\frac{1}{r-2}-\frac{2}{r}+\frac{1}{r+2}\right)$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1999}\right)-2\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{2001}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\cdots+\frac{1}{2003}\right)$$

$$=1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\frac{1}{2000}-\frac{1}{2001}+\frac{1}{2002}+\frac{1}{2003}$$

$$<1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}=\frac{11}{12}$$

2003 Paper 1 Q3

(a) Resolve $\frac{5x-3}{x(x+1)(x+3)}$ into partial fractions.

(b) (i) Prove that $\sum_{k=1}^{n}\frac{5k-3}{k(k+1)(k+3)}<\frac{3}{2}$ for any positive integer n.

(ii) Evaluate $\sum_{k=1}^{\infty}\frac{5k-3}{k(k+1)(k+3)}$.

(a) Let $\frac{5x-3}{x(x+1)(x+3)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x+3}$.

$5x-3 \equiv A(x+1)(x+3)+Bx(x+3)+Cx(x+1)$

Put $x=0 \Rightarrow A=-1$; put $x=-1 \Rightarrow B=4$; put $x=-3 \Rightarrow C=-3$

$\therefore \frac{5x-3}{x(x+1)(x+3)}=-\frac{1}{x}+\frac{4}{x+1}-\frac{3}{x+3}$.

(b) (i)

$$\begin{aligned}\sum_{k=1}^{n}\frac{5k-3}{k(k+1)(k+3)}&=\sum_{k=1}^{n}\left(-\frac{1}{k}+\frac{4}{k+1}-\frac{3}{k+3}\right)\\&=\sum_{k=1}^{n}\left(-\frac{1}{k}+\frac{1}{k+1}\right)+3\sum_{k=1}^{n}\left(\frac{1}{k+1}-\frac{1}{k+3}\right)\\&=\frac{1}{n+1}-1+3\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}\right)\\&=\frac{3}{2}+\frac{1}{n+1}-\frac{3}{n+2}-\frac{3}{n+3}\\&<\frac{3}{2}+\frac{1}{n+1}-\frac{3}{n+2}\quad\left(\because \frac{3}{n+3}>0\right)\\&=\frac{3}{2}-\frac{2n+1}{(n+1)(n+2)}<\frac{3}{2}\end{aligned}$$

(ii) $$\sum_{k=1}^{\infty}\frac{5k-3}{k(k+1)(k+3)}=\frac{3}{2}+\lim_{n\to\infty}\left(\frac{1}{n+1}-\frac{3}{n+2}-\frac{3}{n+3}\right)=\frac{3}{2}$$

2009 Paper 1 Q2

(a) Resolve $\frac{1}{(2x-1)(2x+1)(2x+3)}$ into partial fractions.

(b) Evaluate $\sum_{k=1}^{\infty}\frac{1}{(2k-1)(2k+1)(2k+3)}$.

(c) Find the greatest positive integer m such that $\sum_{k=m}^{\infty}\frac{1}{(2k-1)(2k+1)(2k+3)}>\frac{1}{4000}$.

(a) Let f(x) = $\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right)\left(x+\frac{3}{2}\right)$

$f'(x)=\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right)+\left(x-\frac{1}{2}\right)\left(x+\frac{3}{2}\right)+\left(x+\frac{1}{2}\right)\left(x+\frac{3}{2}\right)$

$f'\left(\frac{1}{2}\right)=2;\ f'\left(-\frac{1}{2}\right)=-1;\ f'\left(-\frac{3}{2}\right)=2$

$$\frac{1}{(2x-1)(2x+1)(2x+3)}=\frac{1}{8}\cdot\frac{1}{\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right)\left(x+\frac{3}{2}\right)}$$

$$=\frac{1}{8}\left[\frac{1}{f'\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)}+\frac{1}{f'\left(-\frac{1}{2}\right)\left(x+\frac{1}{2}\right)}+\frac{1}{f'\left(-\frac{3}{2}\right)\left(x+\frac{3}{2}\right)}\right]$$

$$=\frac{1}{8}\left[\frac{1}{2\left(x-\frac{1}{2}\right)}-\frac{1}{\left(x+\frac{1}{2}\right)}+\frac{1}{2\left(x+\frac{3}{2}\right)}\right]$$

$$=\frac{1}{8(2x-1)}-\frac{1}{4(2x+1)}+\frac{1}{8(2x+3)}$$

(b)

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)(2k+1)(2k+3)}=\sum_{k=1}^{\infty}\left[\frac{1}{8(2k-1)}-\frac{1}{4(2k+1)}+\frac{1}{8(2k+3)}\right]$$

$$=\frac{1}{8}\sum_{k=1}^{\infty}\frac{1}{2k-1}-\frac{1}{8}\sum_{k=1}^{\infty}\frac{2}{2k+1}+\frac{1}{8}\sum_{k=1}^{\infty}\frac{1}{2k+3}$$

$$=\frac{1}{8}\left(1+\frac{1}{3}+\sum_{k=3}^{\infty}\frac{1}{2k-1}\right)-\frac{1}{8}\left(\frac{2}{3}+\sum_{k=2}^{\infty}\frac{2}{2k+1}\right)+\frac{1}{8}\sum_{k=1}^{\infty}\frac{1}{2k+3}$$

$$=\frac{1}{8}\cdot\frac{2}{3}+\frac{1}{8}\sum_{k=1}^{\infty}\frac{1-2+1}{2k+3}=\frac{1}{12}$$

(c) When m = 3, $\sum_{k=2}^{\infty}\frac{1}{(2k-1)(2k+1)(2k+3)}=\frac{1}{12}-\frac{1}{1\times3\times5}-\frac{1}{3\times5\times7}=\frac{1}{140}>\frac{1}{4000}$

For m > 3, $\sum_{k=m}^{\infty}\frac{1}{(2k-1)(2k+1)(2k+3)}>\frac{1}{4000}$

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)(2k+1)(2k+3)}-\sum_{k=1}^{m-1}\frac{1}{(2k-1)(2k+1)(2k+3)}>\frac{1}{4000}$$

$$\frac{1}{12}-\left[\frac{1}{12}-\frac{1}{8}\left(\frac{2}{2(m-1)+1}+\frac{1}{2(m-2)+3}+\frac{1}{2(m-1)+3}\right)\right]>\frac{1}{4000}$$

$$\frac{1}{8}\left(\frac{2}{2m-1}-\frac{1}{2m-1}-\frac{1}{2m+1}\right)>\frac{1}{4000}\Rightarrow\frac{2}{4m^2-1}>\frac{1}{500}\Rightarrow 250.25>m^2$$

m < 15.8

The greatest positive integral m is 15 .

2010 Paper 1 Q2

(a) Resolve $\frac{x}{(x^2-1)(x^2-4)}$ into partial fractions.

(b) By differentiating $\frac{x}{(x^2-1)(x^2-4)}$, or otherwise, resolve $\frac{3x^4-5x^2-4}{(x^2-1)^2(x^2-4)^2}$ into partial fractions.

(c) Evaluate $\sum_{k=3}^{\infty}\frac{3k^4-5k^2-4}{(k^2-1)^2(k^2-4)^2}$.

(a) Let $\frac{x}{(x^2-1)(x^2-4)}\equiv\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x-2}+\frac{D}{x+2}$

$x \equiv A(x+1)(x-2)(x+2)+B(x-1)(x-2)(x+2)+C(x-1)(x+1)(x+2)+D(x-1)(x+1)(x-2)$

Put $x = 1 = -6A \Rightarrow A = -\frac{1}{6}$

Put $x = -1 = 6B \Rightarrow B = -\frac{1}{6}$

Put $x = 2 = 12C \Rightarrow C = \frac{1}{6}$

Put $x = -2 = -12D \Rightarrow D = \frac{1}{6}$

$$\frac{x}{(x^2-1)(x^2-4)}\equiv\frac{1}{6}\left(-\frac{1}{x-1}-\frac{1}{x+1}+\frac{1}{x-2}+\frac{1}{x+2}\right)$$

(b) Differentiate w.r.t. x

$$\frac{(x^2-1)(x^2-4)-x\left[2x(x^2-1)+2x(x^2-4)\right]}{(x^2-1)^2(x^2-4)^2}\equiv\frac{1}{6}\left[\frac{1}{(x-1)^2}+\frac{1}{(x+1)^2}-\frac{1}{(x-2)^2}-\frac{1}{(x+2)^2}\right]$$

$$\frac{x^4-5x^2+4-x(4x^3-10x)}{(x^2-1)^2(x^2-4)^2}\equiv\frac{1}{6}\left[\frac{1}{(x-1)^2}+\frac{1}{(x+1)^2}-\frac{1}{(x-2)^2}-\frac{1}{(x+2)^2}\right]$$

$$\frac{-3x^4+5x^2+4}{(x^2-1)^2(x^2-4)^2}\equiv\frac{1}{6}\left[\frac{1}{(x-1)^2}+\frac{1}{(x+1)^2}-\frac{1}{(x-2)^2}-\frac{1}{(x+2)^2}\right]$$

$$\frac{3x^4-5x^2-4}{(x^2-1)^2(x^2-4)^2}\equiv\frac{1}{6}\left[-\frac{1}{(x-1)^2}-\frac{1}{(x+1)^2}+\frac{1}{(x-2)^2}+\frac{1}{(x+2)^2}\right]$$

(c)
$$\sum_{k=3}^{\infty}\frac{3k^4-5k^2-4}{(k^2-1)^2(k^2-4)^2}=\frac{1}{6}\lim_{n\to\infty}\sum_{k=3}^{n}\left[-\frac{1}{(k-1)^2}-\frac{1}{(k+1)^2}+\frac{1}{(k-2)^2}+\frac{1}{(k+2)^2}\right]$$

$$=\frac{1}{6}\lim_{n\to\infty}\left\{\sum_{k=3}^{n}\left[\frac{1}{(k-2)^2}-\frac{1}{(k-1)^2}\right]+\frac{1}{6}\sum_{k=3}^{n}\left[\frac{1}{(k+2)^2}-\frac{1}{(k+1)^2}\right]\right\}$$

$$=\frac{1}{6}\lim_{n\to\infty}\left[1-\frac{1}{(n-1)^2}+\frac{1}{(n+2)^2}-\frac{1}{16}\right]$$

$$=\frac{1}{6}\cdot\frac{15}{16}$$

$$=\frac{5}{32}$$

2011 Paper 1 Q3

(a) Resolve $\frac{1}{x(x+2)(x+4)}$ into partial fractions.

(b) Let n be a positive integer.

(i) Express $\sum_{k=1}^{n}\frac{1}{k(k+2)(k+4)}$ in the form $A+\frac{B}{n+1}+\frac{C}{n+2}+\frac{D}{n+3}+\frac{E}{n+4}$, where A, B, C, D and E are constants.

(ii) Find $\sum_{k=n+1}^{\infty}\frac{1}{k(k+2)(k+4)}$.

(a) Let $\frac{1}{x(x+2)(x+4)}\equiv\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x+4}$

$1 \equiv A(x+2)(x+4) + Bx(x+4) + Cx(x+2)$

Put $x = 0$: $8A = 1 \Rightarrow A = \frac{1}{8}$

Put $x = -2$: $-4B = 1 \Rightarrow B = -\frac{1}{4}$

Put $x = -4$: $8C = 1 \Rightarrow C = \frac{1}{8}$

$$\frac{1}{x(x+2)(x+4)}\equiv\frac{1}{8x}-\frac{1}{4(x+2)}+\frac{1}{8(x+4)}$$

(b) (i)

$$\sum_{k=1}^{n}\frac{1}{k(k+2)(k+4)}$$

$$=\sum_{k=1}^{n}\left[\frac{1}{8k}-\frac{1}{4(k+2)}+\frac{1}{8(k+4)}\right]$$

$$=\frac{1}{8}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{2}{k+2}+\frac{1}{k+4}\right)$$

$$=\frac{1}{8}\left(\sum_{k=1}^{n}\frac{1}{k}-\sum_{k=1}^{n}\frac{2}{k+2}+\sum_{k=1}^{n}\frac{1}{k+4}\right)$$

$$=\frac{1}{8}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\sum_{k=5}^{n}\frac{1}{k}\right)-\frac{1}{8}\left(\frac{2}{3}+\frac{2}{4}+\sum_{k=3}^{n-2}\frac{2}{k+2}+\frac{2}{n+1}+\frac{2}{n+2}\right)+\frac{1}{8}\left(\sum_{k=1}^{n-4}\frac{1}{k+4}+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\frac{1}{n+4}\right)$$

$$=\frac{25}{96}+\frac{1}{8}\cdot\sum_{k=5}^{n}\frac{1}{k}-\frac{7}{48}-\frac{1}{8}\cdot\sum_{k=5}^{n}\frac{2}{k}-\frac{1}{4(n+1)}-\frac{1}{4(n+2)}+\frac{1}{8}\cdot\sum_{k=5}^{n}\frac{1}{k}+\frac{1}{8}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\frac{1}{n+4}\right)$$

$$=\frac{11}{96}-\frac{1}{8(n+1)}-\frac{1}{8(n+2)}+\frac{1}{8(n+3)}+\frac{1}{8(n+4)}$$

(ii)

$$\sum_{k=n+1}^{\infty}\frac{1}{k(k+2)(k+4)}$$

$$=\sum_{k=1}^{\infty}\frac{1}{k(k+2)(k+4)}-\sum_{k=1}^{n}\frac{1}{k(k+2)(k+4)}$$

$$=\lim_{m\to\infty}\left[\frac{11}{96}-\frac{1}{8(m+1)}-\frac{1}{8(m+2)}+\frac{1}{8(m+3)}+\frac{1}{8(m+4)}\right]-\left[\frac{11}{96}-\frac{1}{8(n+1)}-\frac{1}{8(n+2)}+\frac{1}{8(n+3)}+\frac{1}{8(n+4)}\right]$$

$$=\frac{1}{8(n+1)}+\frac{1}{8(n+2)}-\frac{1}{8(n+3)}-\frac{1}{8(n+4)}=\frac{2n^2+10n+11}{4(n+1)(n+2)(n+3)(n+4)}$$