Supplementary Exercises on Polynomial Identities

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Theorem 1 Let $P(x) \equiv a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_n \neq 0$

If $P(x)$ has more than n different roots, then $P(x) \equiv 0$

Proof: Let $r_1, r_2, \cdots, r_n, r_{n+1}$ be the distinct roots.

then $P(r_1) = 0, P(r_2) = 0, \cdots, P(r_n) = 0, P(r_{n+1}) = 0$

By factor theorem, $P(x) = a_n(x - r_1)(x - r_2) \ldots (x - r_n)(x - r_{n+1})$

If $a_n \neq 0$, then $\deg(P(x)) = n + 1$, which is a contradiction.

$\therefore a_n = 0$ and $P(x) \equiv 0$

Theorem 2 Let $P(x) \equiv a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_n \neq 0$

and $Q(x) \equiv b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$, where $b_n \neq 0$

If $r_1, r_2, \cdots, r_n, r_{n+1}$ are $n + 1$ distinct numbers such that $P(r_k) = Q(r_k)$ for $k = 1, 2, \cdots, n, n + 1$, then $P(x) \equiv Q(x)$

Proof: Let $R(x) = P(x) - Q(x)$, then $\deg(R(x)) \leq n$

$R(r_k) = P(r_k) - Q(r_k) = 0$ for $k = 1, 2, \cdots, n + 1$

Hence $r_1, r_2, \cdots, r_n, r_{n+1}$ are the distinct roots of $R(x)$.

By theorem 1, $R(x) \equiv 0$ and hence $P(x) \equiv Q(x)$.

<u>Exercises</u>

1. By expanding and comparing coefficients, find the values of A, B and C:
   $5x \equiv (Ax + B)(x + 1) + C(x^2 - 2)$
2. If $x^5 + 3x^4 - 6x^3 + 2x^2 - 3x + 7 \equiv A(x + 2)^5 + B(x + 2)^4 + C(x + 2)^3 + D(x + 2)^2 + E(x + 2) + F$
   By differentiating many times and put $x = -2$, find the values of A, B, C, D, E and F.
3. If $4x^4 + 2x^3 + 4x^2 + x + 6 \equiv P(x)(2x^2 + 1)^2 + Q(x)(2x^2 + 1) + R(x)$
   By dividing $(2x^2 + 1)$ many times, find the polynomials $P(x)$, $Q(x)$ and $R(x)$.
4. If $5x^2 - 12x + 9 \equiv (Ax + B)(x - 1)^3 + C_1(x^2 - 2x - 1)(x - 1)^2 + C_2(x^2 - 2x - 1)(x - 1) + C_3(x^2 - 2x - 1)$
   By dividing $(x - 1)$ many times and put $x = 1$, find the values of A, B, C_1, C_2 and C_3.
5. Let a, b, c be three distinct constants. It is given that
   $$\frac{a^2}{(a-b)(a-c)(a+x)} + \frac{b^2}{(b-c)(b-a)(b+x)} + \frac{c^2}{(c-a)(c-b)(c+x)} \equiv \frac{p + qx + rx^2}{(a+x)(b+x)(c+x)}$$
   where p, q r are constants, and $s = 7p + 8q + 9r$, find the value of s.
6. If $axy + bx + cy + d = 0$ can be written in the form $\frac{x-p}{x-q} = \frac{\lambda(y-p)}{y-q}$,
   prove that $\frac{1-\lambda}{a} = \frac{p\lambda - q}{b} = \frac{q\lambda - p}{c} = \frac{pq(1-\lambda)}{d}$,
   where λ satisfies the equation $(ad - bc)(\lambda^2 + 1) = (b^2 + c^2 - 2ad)\lambda$.
7. If the equation $(x - 1)(x^3 - 2x^2 - 7x - 3)$ is written in the form:
   $(x - 2)^4 + p(x - 2)^3 + q(x - 2)^2 + r(x - 2) + s$, find the value of p, q, r and s.
8. If a, b and c are distinct real numbers, prove the identity:
   $$\frac{a^2(x-b)(x-c)}{(a-b)(a-c)} + \frac{b^2(x-c)(x-a)}{(b-c)(b-a)} + \frac{c^2(x-a)(x-b)}{(c-a)(c-b)} \equiv x^2$$
9. If a, b and c are unequal real numbers, prove the identity:
   $$\frac{(x+a)^3}{(a-b)(a-c)} + \frac{(x+b)^3}{(b-c)(b-a)} + \frac{(x+c)^3}{(c-a)(c-b)} \equiv 3x + a + b + c$$

10. If a, b, c and d are unequal real numbers, prove the identity:

$$\frac{(1-bx)(1-cx)(1-dx)}{(a-b)(a-c)(a-d)}+\frac{(1-ax)(1-cx)(1-dx)}{(b-a)(b-c)(b-d)}+\frac{(1-ax)(1-bx)(1-dx)}{(c-a)(c-b)(c-d)}+\frac{(1-ax)(1-bx)(1-cx)}{(d-a)(d-b)(d-c)}\equiv x^3$$

11. Given that p, q and r are distinct values of x which satisfy the equation:

$$\frac{a}{x-\alpha}+\frac{b}{x-\beta}+\frac{c}{x-\gamma}=1$$

(a) Prove that, for all x other than $x = \alpha$, $x = \beta$, $x = \gamma$;

$$\frac{a}{x-\alpha}+\frac{b}{x-\beta}+\frac{c}{x-\gamma}\equiv 1-\frac{(x-p)(x-q)(x-r)}{(x-\alpha)(x-\beta)(x-\gamma)}.$$

(b) Show that $a=\frac{(p-\alpha)(q-\alpha)(r-\alpha)}{(\alpha-\beta)(\alpha-\gamma)}$, $b=\frac{(p-\beta)(q-\beta)(r-\beta)}{(\beta-\alpha)(\beta-\gamma)}$, $c=\frac{(p-\gamma)(q-\gamma)(r-\gamma)}{(\gamma-\alpha)(\gamma-\beta)}$.

(c) Prove that $\frac{a}{(p-\alpha)^2}+\frac{b}{(p-\beta)^2}+\frac{c}{(p-\gamma)^2}\equiv\frac{(p-q)(p-r)}{(p-\alpha)(p-\beta)(p-\gamma)}$.

12. (a) If $ax^2 + 2bx + c$ can be written in the form $A(x-\alpha)^2 + B(x-\beta)^2$,
prove that $a\alpha\beta + b(\alpha+\beta) + c = 0$

(b) Let $u(x) = 16x^2 + 12x + 39$ and $v(x) = 9x^2 - 2x + 11$.
Find the values of λ for which $u(x) + \lambda\, v(x)$ is a perfect square.

(i) Show that $u(x)$ and $v(x)$ can be expressed in the following form:

$u(x) = A(x+\alpha)^2 + B(x+\beta)^2$

$v(x) = A'(x+\alpha)^2 + B'(x+\beta)^2$

for some constants A, A', B, B', α and β.

(ii) Using (b)(i) or otherwise, show that $\frac{3}{2}\le\frac{u(x)}{v(x)}\le 4$ for all x.

13. Prove that if a, b, c are the roots of the equation $x^3 - 3dx - p = 0$,
then $x^6 + px^3 + d^3 \equiv (x^2 + ax + d)(x^2 + bx + d)(x^2 + cx + d)$.
Hence express $x^6 - 20x^3 + 343$ as a product of 3 factors.

End of Exercise

1. $5x \equiv (Ax + B)(x + 1) + C(x^2 - 2)$

$5x \equiv (A + C)x^2 + (A + B)x + (B - 2C)$

Compare coefficients of both sides

x^2: $A + C = 0 \cdots (1)$

x : $A + B = 5 \cdots (2)$

1: $B - 2C = 0 \cdots (3)$

(2) – (3): $A = 5 - 2C \cdots (4)$

Sub. (4) into (1): $5 - 2C + C = 0$

$C = 5, A = -5, B = 10$

2. $x^5 + 3x^4 - 6x^3 + 2x^2 - 3x + 7 \equiv A(x + 2)^5 + B(x + 2)^4 + C(x + 2)^3 + D(x + 2)^2 + E(x + 2) + F$

Put $x = -2 \Rightarrow -32 + 48 + 48 + 8 + 6 + 7 = F \Rightarrow F = 85$

Differentiate once $5x^4 + 12x^3 - 18x^2 + 4x - 3 \equiv 5A(x+2)^4 + 4B(x+2)^3 + 3C(x+2)^2 + 2D(x+2) + E$

Put $x = -2 \Rightarrow 80 - 96 - 72 - 8 - 3 = E \Rightarrow E = -99$

Differentiate twice $20x^3 + 36x^2 - 36x + 4 \equiv 20A(x + 2)^3 + 12B(x + 2)^2 + 6C(x + 2) + 2D$

Put $x = -2 \Rightarrow -160 + 144 + 72 + 4 = 2D \Rightarrow 60 = 2D \Rightarrow D = 30$

Differentiate trice $60x^2 + 72x - 36 \equiv 60A(x + 2)^2 + 24B(x + 2) + 6C$

Put $x = -2 \Rightarrow 240 - 144 - 36 = 6C \Rightarrow 60 = 6C \Rightarrow C = 10$

Differentiate 4 times $120x + 72 \equiv 120A(x + 2) + 24B$

Put $x = -2 \Rightarrow -240 + 72 = 24B \Rightarrow -10 + 3 = B \Rightarrow B = -7$

Differentiate 5 times $120 \equiv 120A \Rightarrow A = 1$

3. $4x^4 + 2x^3 + 4x^2 + x + 6 \equiv P(x)\,(2x^2 + 1)^2 + Q(x)\,(2x^2 + 1) + R(x)$

$$\begin{array}{r|l} & \quad\; 2x^2 + x + 1 \\ 2x^2+1 & 4x^4 + 2x^3 + 4x^2 + x + 6 \\ & -)4x^4 + \quad\;\; 2x^2 \\ \hline & \quad 2x^3 + 2x^2 + x + 6 \\ & -)2x^3 + \qquad\; x \\ \hline & \qquad 2x^2 + \quad 6 \\ & -)2x^2 + \quad 1 \\ \hline & \qquad\qquad 5 \end{array}$$

$$\begin{array}{r|l} & \qquad\quad 1 \\ 2x^2+1 & 2x^2 + x + 1 \\ & -)2x^2 + \quad 1 \\ \hline & \qquad x \end{array}$$

$2x^2 + x + 1 \equiv (2x^2 + 1) + x$

$\therefore 4x^4 + 2x^3 + 4x^2 + x + 6 \equiv (2x^2 + 1)(2x^2 + x + 1) + 5$

$\equiv (2x^2 + 1)^2 + x(2x^2 + 1) + 5$; $P(x) = 1, Q(x) = x, R(x) = 5$

4. $5x^2-12x+9 \equiv (Ax + B)(x-1)^3 + C_1(x^2-2x-1)(x-1)^2 + C_2(x^2-2x-1)(x-1) + C_3(x^2-2x-1) \ldots (1)$

Put $x = 1$, $2 = -2C_3 \Rightarrow C_3 = -1$

(1) becomes $5x^2-12x+9 \equiv (Ax + B)(x-1)^3 + C_1(x^2-2x-1)(x-1)^2 + C_2(x^2-2x-1)(x-1) - (x^2-2x-1)$

$\Rightarrow 5x^2 - 12x + 9 + (x^2 - 2x - 1) \equiv (Ax + B)(x-1)^3 + C_1(x^2-2x-1)(x - 1)^2 + C_2(x^2 - 2x - 1)(x - 1)$

$\Rightarrow 6x^2 - 14x + 8 \equiv (Ax + B)(x - 1)^3 + C_1(x^2 - 2x - 1)(x - 1)^2 + C_2(x^2 - 2x - 1)(x - 1)$

$\Rightarrow 2(x - 1)(3x + 4) \equiv (Ax + B)(x - 1)^3 + C_1(x^2 - 2x - 1)(x - 1)^2 + C_2(x^2 - 2x - 1)(x - 1)$

Divide by $(x - 1) \Rightarrow 6x - 8 \equiv (Ax + B)(x - 1)^2 + C_1(x^2 - 2x - 1)(x - 1) + C_2(x^2 - 2x - 1) \ldots (2)$

Put $x = 1 \Rightarrow -2 = -2C_2 \Rightarrow C_2 = 1$

(2) becomes $6x - 8 \equiv (Ax + B)(x - 1)^2 + C_1(x^2 - 2x - 1)(x - 1) + (x^2 - 2x - 1)$

$6x - 8 - x^2 + 2x + 1 \equiv (Ax + B)(x - 1)^2 + C_1(x^2 - 2x - 1)(x - 1)$

$\Rightarrow -x^2 + 8x - 7 \equiv (Ax + B)(x - 1)^2 + C_1(x^2 - 2x - 1)(x - 1)$

$\Rightarrow -(x - 1)(x - 7) \equiv (Ax + B)(x - 1)^2 + C_1(x^2 - 2x - 1)(x - 1)$

Divide by $(x - 1) \Rightarrow -x + 7 \equiv (Ax + B)(x - 1) + C_1(x^2 - 2x - 1)$ … (3)

Put $x = 1 \Rightarrow 6 = C_1(-2) \Rightarrow C_1 = -3$

(3) becomes $-x + 7 \equiv (Ax + B)(x - 1) - 3(x^2 - 2x - 1)$

$-x + 7 + 3(x^2 - 2x - 1) \equiv (Ax + B)(x - 1)$

$3x^2 - 7x + 4 \equiv (Ax + B)(x - 1)$

$(3x - 4)(x - 1) \equiv (Ax + B)(x - 1)$

$3x - 4 \equiv Ax + B \Rightarrow A = 3, B = -4$

5. $$\frac{a^2}{(a-b)(a-c)(a+x)} + \frac{b^2}{(b-c)(b-a)(b+x)} + \frac{c^2}{(c-a)(c-b)(c+x)} \equiv \frac{p+qx+rx^2}{(a+x)(b+x)(c+x)}$$

Rewrite it as $\dfrac{\frac{a^2}{(a-b)(a-c)}}{(a+x)} + \dfrac{\frac{b^2}{(b-c)(b-a)}}{(b+x)} + \dfrac{\frac{c^2}{(c-a)(c-b)}}{(c+x)} \equiv \dfrac{p+qx+rx^2}{(a+x)(b+x)(c+x)}$

Taking the common denominator, and equating the numerator of both sides.

$$\frac{a^2(b+x)(c+x)}{(a-b)(a-c)} + \frac{b^2(a+x)(c+x)}{(b-c)(b-a)} + \frac{c^2(a+x)(b+x)}{(c-a)(c-b)} \equiv p + qx + rx^2$$

Put $x = -a, -b, -c$ respectively.

$$\begin{cases} a^2 = p - qa + ra^2 \\ b^2 = p - qb + rb^2 \\ c^2 = p - qc + rc^2 \end{cases} \Rightarrow \begin{cases} a^2(r-1) - qa - p = 0 \\ b^2(r-1) - qb - p = 0 \\ c^2(r-1) - qc - p = 0 \end{cases}$$

$\therefore a$, b and c are three distinct roots of $(r - 1)x^2 - qx - p = 0$

By the above theorem, $p = 0$, $q = 0$, $r = 1$.

$s = 7p + 8q + 9r = 9$

6. $axy + bx + cy + d = 0 \cdots (1)$

$$\frac{x-p}{x-q} = \frac{\lambda(y-p)}{y-q}$$

$(x - p)(y - q) = \lambda(x - q)(y - p)$

$xy - py - qx + pq = \lambda xy - \lambda qy - \lambda px + pq\lambda$

$(1 - \lambda)xy + (p\lambda - q)x + (q\lambda - p)y + pq(1 - \lambda) = 0$ … (2)

Compare (1) and (2) $\Rightarrow \dfrac{1-\lambda}{a} = \dfrac{p\lambda - q}{b} = \dfrac{q\lambda - p}{c} = \dfrac{pq(1-\lambda)}{d}$

Let $a = (1 - \lambda)t$, $b = (p\lambda - q)t$, $c = (q\lambda - p)t$, $d = pq(1 - \lambda)t$

To show that $(ad - bc)(\lambda^2 + 1) = (b^2 + c^2 - 2ad)\lambda$.

LHS $= (ad - bc)(\lambda^2 + 1)$

$= [(1 - \lambda)tpq(1 - \lambda)t - (p\lambda - q)t(q\lambda - p)t](\lambda^2 + 1)$

$= [(1 - \lambda)^2 pq - (p\lambda - q)(q\lambda - p)](\lambda^2 + 1)t^2$

$= [pq - 2pq\lambda + pq\lambda^2 - (pq\lambda^2 - q^2\lambda - p^2\lambda + pq)](\lambda^2 + 1)t^2$
$= \lambda(p^2 - 2pq + q^2)(\lambda^2 + 1)t^2$
$= \lambda(\lambda^2 + 1)(p - q)^2 t^2$
RHS $= (b^2 + c^2 - 2ad)\lambda$
$= [(p\lambda - q)^2 t^2 + (q\lambda - p)^2 t^2 - 2(1 - \lambda)tpq(1 - \lambda)t]\lambda$
$= \lambda[(p\lambda - q)^2 + (q\lambda - p)^2 - 2pq(1 - \lambda)^2]t^2$
$= \lambda(p^2\lambda^2 - 2pq\lambda + q^2 + q^2\lambda^2 - 2pq\lambda + p^2 - 2pq + 4pq\lambda - 2pq\lambda^2)\, t^2$
$= \lambda(p^2\lambda^2 - 2pq\lambda^2 + q^2\lambda^2 + p^2 - 2pq + q^2)\, t^2$
$= \lambda[\lambda^2 (p - q)^2 + (p - q)^2]\, t^2$
$= \lambda(\lambda^2 + 1)(p - q)^2 t^2$
$\therefore$ LHS = RHS

7. $(x - 1)(x^3 - 2x^2 - 7x - 3) = x^4 - 3x^3 - 5x^2 + 4x + 3$
$x^4-3x^3-5x^2+4x+3 \equiv (x-2)^4+p(x-2)^3+q(x-2)^2+r(x-2)+s$
Put $x = 2$: $(2 - 1)(8 - 8 - 14 - 3) = s \Rightarrow s = -17$
Differentiate once:
$4x^3 - 9x^2 - 10x + 4 \equiv 4(x-2)^3 + 3p(x-2)^2 + 2q(x-2)+r$

		1	−2	−7	−3
×)				1	−1
	1	−2	−7	−3	
		−1	2	7	3
	1	−3	−5	4	3

Put $x = 2$: $32 - 36 - 20 + 4 = r \Rightarrow r = -20$
Differentiate twice: $12x^2 - 18x - 10 \equiv 12(x - 2)^2 + 6p(x - 2) + 2q$
Put $x = 2$: $48 - 36 - 10 = 2q \Rightarrow q = 1$
Differentiate trice: $24x - 18 \equiv 24(x - 2) + 6p$
Put $x = 2$: $48 - 18 = 6p \Rightarrow p = 5$

8. $\frac{a^2(x-b)(x-c)}{(a-b)(a-c)} + \frac{b^2(x-c)(x-a)}{(b-c)(b-a)} + \frac{c^2(x-a)(x-b)}{(c-a)(c-b)} \equiv x^2$
Put $x = a$, LHS $= a^2$, RHS $= a^2$
Put $x = b$, LHS $= b^2$, RHS $= b^2$
Put $x = c$, LHS $= c^2$, RHS $= c^2$
$\because$ LHS is a polynomial in x with degree 2, so is RHS.
$\therefore$ LHS $\equiv$ RHS

9. $\frac{(x+a)^3}{(a-b)(a-c)} + \frac{(x+b)^3}{(b-c)(b-a)} + \frac{(x+c)^3}{(c-a)(c-b)} \equiv 3x + a + b + c$
Put $x = -a$, LHS $= \frac{(b-a)^2}{(b-c)} + \frac{(c-a)^2}{(c-b)}$; RHS $= -3a + a + b + c = -2a + b + c$
$= \frac{(b-a)^2 - (c-a)^2}{b-c}$
$= \frac{(b-a+c-a)(b-a-c+a)}{b-c}$
$= \frac{(b+c-2a)(b-c)}{b-c}$
$= b + c - 2a =$ RHS
$\because$ The expression are cyclic in a, b and c. $\therefore$ When $x = -b$ or $-c$, LHS = RHS
Compare coefficient of x^3:
LHS $= \frac{1}{(a-b)(a-c)} + \frac{1}{(b-c)(b-a)} + \frac{1}{(c-a)(c-b)}$; RHS = 0
$= \frac{c-b+a-c+b-a}{(a-b)(b-c)(c-a)} = 0 =$ RHS

LHS is a polynomial in x of degree < 3, RHS is a polynomial of degree 1.

$\therefore$ LHS $\equiv$ RHS

10. $\frac{(1-bx)(1-cx)(1-dx)}{(a-b)(a-c)(a-d)}+\frac{(1-ax)(1-cx)(1-dx)}{(b-a)(b-c)(b-d)}+\frac{(1-ax)(1-bx)(1-dx)}{(c-a)(c-b)(c-d)}+\frac{(1-ax)(1-bx)(1-cx)}{(d-a)(d-b)(d-c)}\equiv x^3$

If a, b, c, $d\neq 0$, put $x=\frac{1}{a}$, LHS $=\frac{1}{a^3}\frac{(a-b)(a-c)(a-d)}{(a-b)(a-c)(a-d)}=\frac{1}{a^3}=$ RHS

Similarly, put $x=\frac{1}{b}$, $\frac{1}{c}$, $\frac{1}{d}$; LHS = RHS

LHS is a polynomial in x of degree 3, so is RHS.

$\therefore$ LHS = RHS

If $a = 0$, then b, c, $d \neq 0$,

LHS $=\frac{(1-bx)(1-cx)(1-dx)}{-bcd}+\frac{(1-cx)(1-dx)}{b(b-c)(b-d)}+\frac{(1-bx)(1-dx)}{c(c-b)(c-d)}+\frac{(1-bx)(1-cx)}{d(d-b)(d-c)}$

Put $x=\frac{1}{b}$, LHS $=\frac{1}{b^3}=$ RHS

Similarly, put $x=\frac{1}{c}$, $\frac{1}{d}$, LHS = RHS.

Compare the coefficient of x^3: LHS = 1 = RHS

$\therefore$ LHS $\equiv$ RHS

11. $\frac{a}{x-\alpha}+\frac{b}{x-\beta}+\frac{c}{x-\gamma}=1$, roots: p, q and r.

$a(x-\beta)(x-\gamma)+b(x-\alpha)(x-\gamma)+c(x-\alpha)(x-\beta)=(x-\alpha)(x-\beta)(x-\gamma)$ … (1)

Put $x = p \Rightarrow a(p-\beta)(p-\gamma)+b(p-\alpha)(p-\gamma)+c(p-\alpha)(p-\beta)=(p-\alpha)(p-\beta)(p-\gamma)$ … (2)

(a) To prove that $\frac{a}{x-\alpha}+\frac{b}{x-\beta}+\frac{c}{x-\gamma}\equiv 1-\frac{(x-p)(x-q)(x-r)}{(x-\alpha)(x-\beta)(x-\gamma)}$ … (3)

$\Leftrightarrow \frac{a(x-\beta)(x-\gamma)+b(x-\alpha)(x-\gamma)+c(x-\alpha)(x-\beta)}{(x-\alpha)(x-\beta)(x-\gamma)}\equiv\frac{(x-\alpha)(x-\beta)(x-\gamma)-(x-p)(x-q)(x-r)}{(x-\alpha)(x-\beta)(x-\gamma)}$ … (4)

Put $x = p$, LHS $=\frac{a(p-\beta)(p-\gamma)+b(p-\alpha)(p-\gamma)+c(p-\alpha)(p-\beta)}{(p-\alpha)(p-\beta)(p-\gamma)}$

$=\frac{(p-\alpha)(p-\beta)(p-\gamma)}{(p-\alpha)(p-\beta)(p-\gamma)}=1$ (by (2))

RHS $=\frac{(p-\alpha)(p-\beta)(p-\gamma)}{(p-\alpha)(p-\beta)(p-\gamma)}=1$; $\therefore$LHS = RHS

Similarly, put $x = q$ and $x = r$, LHS = RHS

Numerator of LHS of (4) is a polynomial of degree 2.

Numerator of RHS of (4) is also a polynomial of degree 2.

$\therefore$ LHS $\equiv$ RHS

(b) Compare the numerator of (4):

$a(x-\beta)(x-\gamma)+b(x-\alpha)(x-\gamma)+c(x-\alpha)(x-\beta)\equiv(x-\alpha)(x-\beta)(x-\gamma)-(x-p)(x-q)(x-r)$

Put $x = \alpha \Rightarrow a(\alpha-\beta)(\alpha-\gamma)=-(\alpha-p)(\alpha-q)(\alpha-r)$

$\Rightarrow a=\frac{(p-\alpha)(q-\alpha)(r-\alpha)}{(\alpha-\beta)(\alpha-\gamma)}$

Similarly, put $x = \beta$, $b=\frac{(p-\beta)(q-\beta)(r-\beta)}{(\beta-\alpha)(\beta-\gamma)}$; put $x = \gamma$, $c=\frac{(p-\gamma)(q-\gamma)(r-\gamma)}{(\gamma-\alpha)(\gamma-\beta)}$.

(c) Differentiate (3): $\frac{a}{x-\alpha}+\frac{b}{x-\beta}+\frac{c}{x-\gamma}\equiv 1-\frac{(x-p)(x-q)(x-r)}{(x-\alpha)(x-\beta)(x-\gamma)}$

$$-\frac{a}{(x-\alpha)^2}-\frac{b}{(x-\beta)^2}-\frac{c}{(x-\gamma)^2}\equiv-\frac{d}{dx}\left[\frac{(x-p)(x-q)(x-r)}{(x-\alpha)(x-\beta)(x-\gamma)}\right] \quad \cdots\cdots (5)$$

Let $y=\frac{(x-p)(x-q)(x-r)}{(x-\alpha)(x-\beta)(x-\gamma)}$

$\ln y = \ln(x-p)+\ln(x-q)+\ln(x-r)-\ln(x-\alpha)-\ln(x-\beta)-\ln(x-\gamma)$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{x-p}+\frac{1}{x-q}+\frac{1}{x-r}-\frac{1}{x-\alpha}-\frac{1}{x-\beta}-\frac{1}{x-\gamma}$$

$$\frac{dy}{dx}=y\left[\frac{1}{x-p}+\frac{1}{x-q}+\frac{1}{x-r}-\frac{1}{x-\alpha}-\frac{1}{x-\beta}-\frac{1}{x-\gamma}\right]$$

$$=\frac{(x-q)(x-r)}{(x-\alpha)(x-\beta)(x-\gamma)}+\frac{(x-p)(x-q)(x-r)}{(x-\alpha)(x-\beta)(x-\gamma)}\left(\frac{1}{x-q}+\frac{1}{x-r}-\frac{1}{x-\alpha}-\frac{1}{x-\beta}-\frac{1}{x-\gamma}\right)$$

Sub. into (5):

$$-\frac{a}{(x-\alpha)^2}-\frac{b}{(x-\beta)^2}-\frac{c}{(x-\gamma)^2}\equiv-\frac{(x-q)(x-r)}{(x-\alpha)(x-\beta)(x-\gamma)}-\frac{(x-p)(x-q)(x-r)}{(x-\alpha)(x-\beta)(x-\gamma)}\left(\frac{1}{x-q}+\frac{1}{x-r}-\frac{1}{x-\alpha}-\frac{1}{x-\beta}-\frac{1}{x-\gamma}\right)$$

Put $x = p$ and multiply by -1, $\frac{a}{(p-\alpha)^2}+\frac{b}{(p-\beta)^2}+\frac{c}{(p-\gamma)^2}\equiv\frac{(p-q)(p-r)}{(p-\alpha)(p-\beta)(p-\gamma)}$.

12. (a) $ax^2 + 2bx + c \equiv A(x-\alpha)^2 + B(x-\beta)^2$

Put $x = \alpha \Rightarrow a\alpha^2 + 2b\alpha + c \equiv B(\alpha-\beta)^2$ ⋯⋯ (1)

Put $x = \beta \Rightarrow a\beta^2 + 2b\beta + c \equiv A(\beta-\alpha)^2$ ⋯ (2)

Compare coefficient of x^2: $a = A + B$ ⋯ (3)

(1) + (2): $a(\alpha^2+\beta^2) + 2b(\alpha+\beta) + 2c = (A+B)(\alpha-\beta)^2$

By (3): $a(\alpha^2+\beta^2) + 2b(\alpha+\beta) + 2c = a(\alpha^2-2\alpha\beta+\beta^2)$

$\Rightarrow 2a(\alpha\beta) + 2b(\alpha+\beta) + 2c = 0$

$\Rightarrow a\alpha\beta + b(\alpha+\beta) + c = 0$

(b) $u(x) = 16x^2 + 12x + 39$, $v(x) = 9x^2 - 2x + 11$.

$u(x) + \lambda\, v(x) = 16x^2 + 12x + 39 + \lambda(9x^2 - 2x + 11)$

$= (16 + 9\lambda)x^2 + (12 - 2\lambda)x + 39 + 11\lambda$

For $u(x) + \lambda\, v(x)$ to be a perfect square, $u(x) + \lambda\, v(x) = 0$ has a double root $\Rightarrow \Delta = 0$

$\Delta = (12 - 2\lambda)^2 - 4(16 + 9\lambda)(39 + 11\lambda) = 0$

$4[(6 - \lambda)^2 - (16 + 9\lambda)(39 + 11\lambda)] = 0$

$36 - 12\lambda + \lambda^2 - 624 - 351\lambda - 176\lambda - 99\lambda^2 = 0$

$-98\lambda^2 - 539\lambda - 588 = 0$

$2\lambda^2 + 11\lambda + 12 = 0$

$(\lambda + 4)(2\lambda + 3) = 0$

$\Rightarrow \lambda = -4$ or $\lambda = -\frac{3}{2}$

(i) When $\lambda = -4$, $u(x) - 4\,v(x) = -20x^2 + 20x - 5 = -20\left(x-\frac{1}{2}\right)^2$ … (4)

When $\lambda=-\frac{3}{2}$, $u(x)-\frac{3}{2}v(x)=\frac{5}{2}x^2+\frac{30}{2}x+\frac{45}{2}=\frac{5}{2}(x+3)^2$ … (5)

(5) – (4): $\left(4-\frac{3}{2}\right)v(x)=20\left(x-\frac{1}{2}\right)^2+\frac{5}{2}(x+3)^2$

$\Rightarrow \frac{5}{2}v(x)=20\left(x-\frac{1}{2}\right)^2+\frac{5}{2}(x+3)^2$

$\Rightarrow v(x) = 8\left(x-\frac{1}{2}\right)^2 + (x+3)^2$ … (6)

Sub. (6) into (4): $u(x) = 4\,v(x) - 20\left(x-\frac{1}{2}\right)^2$

$= 32\left(x-\frac{1}{2}\right)^2 + 4(x+3)^2 - 20\left(x-\frac{1}{2}\right)^2$

$= 12\left(x-\frac{1}{2}\right)^2 + 4(x+3)^2$

(ii) $\frac{u(x)}{v(x)} = \frac{12\left(x-\frac{1}{2}\right)^2 + 4(x+3)^2}{8\left(x-\frac{1}{2}\right)^2 + (x+3)^2}$

From (4): $u(x) - 4\,v(x) = -20\left(x-\frac{1}{2}\right)^2 \le 0$

From (5): $u(x) - \frac{3}{2}v(x) \ge 0$

$\therefore\ -\frac{3}{2} \le \frac{u(x)}{v(x)} \le 4$

13. $x^3 - 3dx - p = 0$, roots: a, b, c

By the relation between the roots and coefficients:

$a + b + c = 0$, $ab + bc + ca = -3d$, $abc = p$

To prove that $x^6 + px^3 + d^3 \equiv (x^2 + ax + d)(x^2 + bx + d)(x^2 + cx + d)$.

Let $t = x^2 + d$, RHS $= (t + ax)(t + bx)(t + cx)$

$= t^3 + (a + b + c)xt^2 + (ab + bc + ca)x^2t + abcx^3$

$= t^3 - 3dx^2t + px^3$

$= (x^2 + d)^3 - 3dx^2(x^2 + d) + px^3$

$= x^6 + 3dx^4 + 3x^2d^2 + d^3 - 3dx^4 - 3d^2x^2 + px^3$

$= x^6 + px^3 + d^3 =$ LHS

Compare $x^6 - 20x^3 + 343$ with $x^6 + px^3 + d^3$:

$p = -20$, $d^3 = 343 \Rightarrow d = 7$

$$\begin{cases} a+b+c=0 \cdots\cdots(1) \\ ab+bc+ca=-21\cdots\cdots(2) \\ abc=-20\cdots\cdots(3) \end{cases}$$

By guess, $a = 1$, $b = 4$, $c = -5$

Sub. into (1): $1 + 4 - 5 = 0$

Sub. into (2): $4 - 20 - 5 = -21$

Sub. into (3): $1(4)(-5) = -20$

$\therefore\ x^6 - 20x^3 + 343 \equiv (x^2 + x + 7)(x^2 + 4x + 7)(x^2 - 5x + 7)$