20th Moscow University Mathematics Olympiad (1957) Q29

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Solve the system of equations $\begin{cases} \dfrac{2x_1^2}{1+x_1^2} = x_2 \\ \dfrac{2x_2^2}{1+x_2^2} = x_3 \\ \dfrac{2x_3^2}{1+x_3^2} = x_1 \end{cases}$.

If $x_1 = 0$, then $x_2 = 0$ and $x_3 = 0$. Hence (0, 0, 0) is a solution. Otherwise, the system is equivalent to

$$\begin{cases} \dfrac{1+x_1^2}{2x_1^2} = \dfrac{1}{x_2} \\ \dfrac{1+x_2^2}{2x_2^2} = \dfrac{1}{x_3} \\ \dfrac{1+x_3^2}{2x_3^2} = \dfrac{1}{x_1} \end{cases} \Rightarrow \begin{cases} 1+\dfrac{1}{x_1^2} = \dfrac{2}{x_2} \\ 1+\dfrac{1}{x_2^2} = \dfrac{2}{x_3} \\ 1+\dfrac{1}{x_3^2} = \dfrac{2}{x_1} \end{cases} \Rightarrow \text{Let } a = \dfrac{1}{x_1},\ b = \dfrac{1}{x_2},\ c = \dfrac{1}{x_3}, \text{ then } \begin{cases} 1+a^2 = 2b \\ 1+b^2 = 2c \\ 1+c^2 = 2a \end{cases}.$$

Add these 3 equations together: $3 + a^2 + b^2 + c^2 = 2b + 2c + 2a$

$(a^2 - 2a + 1) + (b^2 - 2b + 1) + (c^2 - 2c + 1) = 0$

$(a - 1)^2 + (b - 1)^2 + (c - 1)^2 = 0$

Sum of 3 squares = 0 $\Rightarrow$ each term = 0

$a = 1$, $b = 1$ and $c = 1$

The solutions are $(x, y, z) = (0, 0, 0)$ or $(1, 1, 1)$.