Factorization

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Factorization of $z^{2n} - 2a^n z^n \cos n\theta + a^{2n}$, a is a real number and $n > 1$.

Let $f(z) = z^{2n} - 2a^n z^n \cos n\theta + a^{2n}$; consider the roots of $f(z) = 0$

$(z^n)^2 - (2a^n \cos n\theta)z^n + a^{2n} = 0$, this is a quadratic equation in z^n.

$$z^n = \frac{2a^n \cos n\theta \pm \sqrt{4\cos^2 n\theta a^{2n} - 4a^{2n}}}{2}$$

$$z^n = a^n \cos n\theta \pm \sqrt{\cos^2 n\theta a^{2n} - a^{2n}}$$

$$z^n = a^n\left(\cos n\theta \pm \sqrt{\cos^2 n\theta - 1}\right)$$

$$z^n = a^n\left(\cos n\theta \pm \sqrt{-\sin^2 n\theta}\right)$$

$$z^n = a^n\left(\cos n\theta \pm \sin n\theta\sqrt{-1}\right)$$

$z^n = a^n(\cos n\theta \pm i \sin n\theta)$, where $i = \sqrt{-1}$.

$z^n = a^n(\cos n\theta + i \sin n\theta)$, or $z^n = a^n[\cos(-n\theta) + i\sin(-n\theta)]$

$z^n = a^n[\cos(n\theta + 2k\pi) + i\sin(n\theta + 2k\pi)]$ or $a^n[\cos(-n\theta - 2k\pi) + i\sin(-n\theta - 2k\pi)]$

where $k = 0, 1, 2, 3, \ldots, (n - 1)$.

$$z = a\left(\cos\frac{n\theta + 2k\pi}{n} + i\sin\frac{n\theta + 2k\pi}{n}\right) \text{ or } z = a\left(\cos\frac{-n\theta - 2k\pi}{n} + i\sin\frac{-n\theta - 2k\pi}{n}\right)$$

$$z = a\left[\cos\left(\theta + \frac{2k\pi}{n}\right) + i\sin\left(\theta + \frac{2k\pi}{n}\right)\right] \text{ or } z = a\left[\cos\left(\theta + \frac{2k\pi}{n}\right) - i\sin\left(\theta + \frac{2k\pi}{n}\right)\right]$$

$\therefore$ Factors of $f(z)$ are: (for $k = 0, 1, 2, 3, \ldots, (n - 1)$.)

$$z - a\left[\cos\left(\theta + \frac{2k\pi}{n}\right) + i\sin\left(\theta + \frac{2k\pi}{n}\right)\right] \text{ and } z - a\left[\cos\left(\theta + \frac{2k\pi}{n}\right) - i\sin\left(\theta + \frac{2k\pi}{n}\right)\right]$$

Quadratic factors are: (for $k = 0, 1, 2, 3, \ldots, (n - 1)$.)

$$\left\{z - a\left[\cos\left(\theta + \frac{2k\pi}{n}\right) + i\sin\left(\theta + \frac{2k\pi}{n}\right)\right]\right\}\left\{z - a\left[\cos\left(\theta + \frac{2k\pi}{n}\right) - i\sin\left(\theta + \frac{2k\pi}{n}\right)\right]\right\}$$

$$= \left[z - a\cos\left(\theta + \frac{2k\pi}{n}\right) - ia\sin\left(\theta + \frac{2k\pi}{n}\right)\right]\left[z - a\cos\left(\theta + \frac{2k\pi}{n}\right) + ia\sin\left(\theta + \frac{2k\pi}{n}\right)\right]$$

$$= \left[z - a\cos\left(\theta + \frac{2k\pi}{n}\right)\right]^2 - \left[ia\sin\left(\theta + \frac{2k\pi}{n}\right)\right]^2$$

$$= z^2 - 2az\cos\left(\theta + \frac{2k\pi}{n}\right) + a^2\cos^2\left(\theta + \frac{2k\pi}{n}\right) + a^2\sin^2\left(\theta + \frac{2k\pi}{n}\right)$$

$$= z^2 - 2az\cos\left(\theta + \frac{2k\pi}{n}\right) + a^2; \text{ for } k = 0, 1, 2, 3, \ldots, (n - 1).$$

$$z^{2n} - 2a^n z^n \cos n\theta + a^{2n}$$

$$= \left(z^2 - 2az\cos\theta + a^2\right)\left[z^2 - 2az\cos\left(\theta + \frac{2\pi}{n}\right) + a^2\right]\cdots\left\{z^2 - 2az\cos\left[\theta + \frac{2(n-1)\pi}{n}\right] + a^2\right\}$$

Special Cases

(1) $\theta = 0$, $a = 1$

$$z^{2n} - 2z^n + 1 = (z^2 - 2z + 1)\left(z^2 - 2z\cos\frac{2\pi}{n} + 1\right)\cdots\left[z^2 - 2z\cos\frac{2(n-1)\pi}{n} + 1\right]$$

$$\frac{(z^n - 1)^2}{(z-1)^2} = \left(z^2 - 2z\cos\frac{2\pi}{n} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{n} + 1\right)\cdots\left[z^2 - 2z\cos\frac{2(n-1)\pi}{n} + 1\right],\ z \neq 1$$

$$(1 + z + z^2 + \cdots + z^{n-1})^2 = \left(z^2 - 2z\cos\frac{2\pi}{n} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{n} + 1\right)\cdots\left[z^2 - 2z\cos\frac{2(n-1)\pi}{n} + 1\right]$$

As $z \to 1$, $n^2 = \left(2 - 2\cos\frac{2\pi}{n}\right)\left(2 - 2\cos\frac{4\pi}{n}\right)\cdots\left[2 - 2\cos\frac{2(n-1)\pi}{n}\right]$

$$n^2 = 2^{n-1}\left(1 - \cos\frac{2\pi}{n}\right)\left(1 - \cos\frac{4\pi}{n}\right)\cdots\left[1 - \cos\frac{2(n-1)\pi}{n}\right]$$

$\frac{n^2}{2^{n-1}} = \left(2\sin^2\frac{\pi}{n}\right)\left(2\sin^2\frac{2\pi}{n}\right)\cdots\left[2\sin^2\frac{(n-1)\pi}{n}\right]$, using the identity $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$

$$\frac{n^2}{2^{n-1}} = 2^{n-1}\left[\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n}\right]^2$$

$$\left(\frac{n}{2^{n-1}}\right)^2 = \left[\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n}\right]^2$$

$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$, $\because \sin\frac{k\pi}{n} > 0$ for $k = 1, 2, \ldots, (n-1)$

$\because \sin\frac{(n-k)\pi}{n} = \sin\frac{k\pi}{n}$

When $n = 2m$, $\left[\sin\frac{\pi}{2m}\sin\frac{2\pi}{2m}\cdots\sin\frac{(m-1)\pi}{2m}\right]^2 = \frac{2m}{2^{2m-1}}$

$$\sin\frac{\pi}{2m}\sin\frac{2\pi}{2m}\cdots\sin\frac{(m-1)\pi}{2m} = \sqrt{\frac{2m}{2^{2m-1}}} = \frac{\sqrt{m}}{2^{m-1}}$$

When $n = 2m + 1$, $\left(\sin\frac{\pi}{2m+1}\sin\frac{2\pi}{2m+1}\cdots\sin\frac{m\pi}{2m+1}\right)^2 = \frac{2m+1}{2^{2m}}$

$$\sin\frac{\pi}{2m+1}\sin\frac{2\pi}{2m+1}\cdots\sin\frac{m\pi}{2m+1} = \sqrt{\frac{2m+1}{2^{2m}}} = \frac{\sqrt{2m+1}}{2^m}$$

$n = 5 \Rightarrow m = 2 \Rightarrow \sin 36^\circ \sin 72^\circ = \frac{\sqrt{5}}{4}$ ....(1)

$n = 15 \Rightarrow m = 7 \Rightarrow \sin 12^\circ \sin 24^\circ \sin 36^\circ \sin 48^\circ \sin 60^\circ \sin 72^\circ \sin 84^\circ = \frac{\sqrt{15}}{128}$ ....(2)

(2) ÷ (1) $\sin 12^\circ \sin 24^\circ \sin 48^\circ \sin 60^\circ \sin 84^\circ = \frac{\sqrt{3}}{32}$

$$\frac{\sqrt{3}}{2}\sin 12^\circ \sin 24^\circ \sin 48^\circ \sin 84^\circ = \frac{\sqrt{3}}{32}$$

$$\sin 12^\circ \sin 24^\circ \sin 48^\circ \sin 84^\circ = \frac{1}{16}$$

$\sin 12^\circ \sin 24^\circ \sin 48^\circ \sin 96^\circ = \frac{1}{16}$ $\because \sin 84^\circ = \sin 96^\circ$

(2) $z = \cos\phi + i\sin\phi$, $a = 1$

then $z + \frac{1}{z} = 2\cos\phi$; $z^n + \frac{1}{z^n} = 2\cos n\phi$

$$z^{2n} - 2z^n \cos n\theta + 1$$

$$= \left(z^2 - 2z\cos\theta + 1\right)\left[z^2 - 2z\cos\left(\theta + \frac{2\pi}{n}\right) + 1\right]\cdots\left\{z^2 - 2z\cos\left[\theta + \frac{2(n-1)\pi}{n}\right] + 1\right\}$$

Divide both sides by z^n

$$z^n - 2\cos n\theta + \frac{1}{z^n}$$

$$= \left(z - 2\cos\theta + \frac{1}{z}\right)\left[z - 2\cos\left(\theta + \frac{2\pi}{n}\right) + \frac{1}{z}\right]\cdots\left\{z - 2\cos\left[\theta + \frac{2(n-1)\pi}{n}\right] + \frac{1}{z}\right\}$$

$$2\cos n\phi - 2\cos n\theta$$

$$= \left(2\cos\phi - 2\cos\theta\right)\left[2\cos\phi - 2\cos\left(\theta + \frac{2\pi}{n}\right)\right]\cdots\left\{2\cos\phi - 2\cos\left[\theta + \frac{2(n-1)\pi}{n}\right]\right\}$$

$$\cos n\phi - \cos n\theta = 2^{n-1}\left(\cos\phi - \cos\theta\right)\left[\cos\phi - \cos\left(\theta + \frac{2\pi}{n}\right)\right]\cdots\left\{\cos\phi - \cos\left[\theta + \frac{2(n-1)\pi}{n}\right]\right\}$$

(3) $z = a = 1$, $\theta = 2\beta$ with $0 < \beta < \frac{\pi}{n}$.

$$1 - 2\cos 2n\beta + 1 = \left(1 - 2\cos 2\beta + 1\right)\left[1 - 2\cos\left(2\beta + \frac{2\pi}{n}\right) + 1\right]\cdots\left\{1 - 2\cos\left[2\beta + \frac{2(n-1)\pi}{n}\right] + 1\right\}$$

$$2(1 - \cos 2n\beta) = 2^n\left(1 - \cos 2\beta\right)\left[1 - \cos\left(2\beta + \frac{2\pi}{n}\right)\right]\cdots\left\{1 - \cos\left[2\beta + \frac{2(n-1)\pi}{n}\right]\right\}$$

$$2^2\sin^2 n\beta = \left(2^n\right)^2 \sin^2\beta \sin^2\left(\beta + \frac{\pi}{n}\right)\cdots\sin^2\left[\beta + \frac{(n-1)\pi}{n}\right]$$

$$\sin n\beta = \pm 2^{n-1}\sin\beta\sin\left(\beta + \frac{\pi}{n}\right)\cdots\sin\left[\beta + \frac{(n-1)\pi}{n}\right]$$

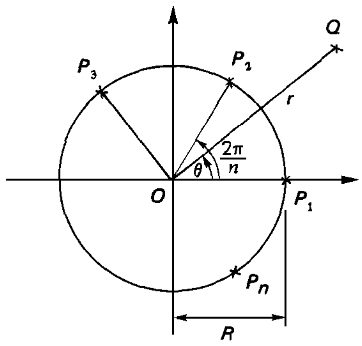
As $0 < n\beta < \pi$ and $0 < \beta + \frac{k\pi}{n} < \pi$, $\sin n\beta > 0$ and each factor on the right is positive. Hence the ambiguous sign $\pm$ is positive.

$$\sin n\beta = 2^{n-1}\sin\beta\sin\left(\beta + \frac{\pi}{n}\right)\cdots\sin\left[\beta + \frac{(n-1)\pi}{n}\right]$$

(4) De Moivre's Property of Circle.

The figure shows a circle with centre O and radius R. P_1, P_2, … P_n are points on the circle dividing the circle into equal arcs with P_1 lies on the horizontal axis. Q is a point such that $OQ = r$ and $\angle P_1OQ = \theta$.

By cosine formula,



$$P_1Q = \sqrt{R^2 - 2Rr\cos\theta + r^2}$$

$$P_2Q = \sqrt{R^2 - 2Rr\cos\left(\frac{2\pi}{n} - \theta\right) + r^2}$$

$$P_3Q = \sqrt{R^2 - 2Rr\cos\left(\frac{4\pi}{n} - \theta\right) + r^2}$$

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$$P_nQ = \sqrt{R^2 - 2Rr\cos\left[\frac{2(n-1)\pi}{n} - \theta\right] + r^2}$$

$$P_1Q \cdot P_2Q \cdots P_nQ = \sqrt{\left(R^2 - 2Rr\cos\theta + r^2\right)\left[R^2 - 2Rr\cos\left(\frac{2\pi}{n} - \theta\right) + r^2\right]\cdots\left\{R^2 - 2Rr\cos\left[\frac{2(n-1)\pi}{n} - \theta\right] + r^2\right\}}$$

$$= \sqrt{\left(R^2 - 2Rr\cos\theta + r^2\right)\left[R^2 - 2Rr\cos\left(\theta + \frac{2\pi}{n}\right) + r^2\right]\cdots\left\{R^2 - 2Rr\cos\left[\theta + \frac{2(n-1)\pi}{n}\right] + r^2\right\}}$$

$$\because\ \cos\left[\frac{2(n-1)\pi}{n} - \theta\right] = \cos\left(\theta + \frac{2\pi}{n}\right); \cos\left(\frac{2\pi}{n} - \theta\right) = \cos\left[\theta + \frac{2(n-1)\pi}{n}\right]$$

By putting $z = R$, $a = r$ into the original factorization,

$$P_1Q \cdot P_2Q \cdots P_nQ = \sqrt{R^{2n} - 2R^n r^n \cos n\theta + r^{2n}}$$