# Complex Number Hard example

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(a) Solve $z^5 - 1 = 0$

(b) Hence, or otherwise, show that

$(y + 1)^5 - (y - 1)^5 = 10\left(y^2 + \cot^2\frac{\pi}{5}\right)\left(y^2 + \cot^2\frac{2\pi}{5}\right)$

(c) Hence, deduce that

(i) $\csc^2\frac{\pi}{5} + \csc^2\frac{2\pi}{5} = 4$

(ii) $\tan\frac{\pi}{5} - \tan\frac{2\pi}{5} = -\sqrt{10 - 2\sqrt{5}}$

(a) $z^5 - 1 = 0$

$z^5 = \cos 2k\pi + i\sin 2k\pi$, $k = -2, -1, 0, 1, 2$

$z = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$, $k = -2, -1, 0, 1, 2$

$z = 1$, $\cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$, $k = -2, -1, 1, 2$

(b) $z^5 - 1$

$$= (z-1)\left(z - \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)\left(z - \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}\right)\left(z - \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)\left(z - \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}\right)$$

$$= (z-1)\left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right) \quad \cdots\cdots (*)$$

Divide both sides by $z - 1$: $\frac{z^5 - 1}{z - 1} = \left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right)$

$$z^4 + z^3 + z^2 + z + 1 = \left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right)$$

As $z \to 1$: $\left(1 - 2\cos\frac{2\pi}{5} + 1\right)\left(1 - 2\cos\frac{4\pi}{5} + 1\right) = 1 + 1 + 1 + 1 + 1$

$$4\left(1 - \cos\frac{2\pi}{5}\right)\left(1 - \cos\frac{4\pi}{5}\right) = 5$$

$$\left(1 - 1 + 2\sin^2\frac{\pi}{5}\right)\left(1 - 1 + 2\sin^2\frac{2\pi}{5}\right) = \frac{5}{4}$$

$$\left(\sin\frac{\pi}{5}\right)^2\left(\sin\frac{2\pi}{5}\right)^2 = \frac{5}{16} \quad \cdots\cdots (**)$$

Put $z = \frac{y+1}{y-1}$ into (*)

$$\left(\frac{y+1}{y-1}\right)^5 - 1 = \left(\frac{y+1}{y-1} - 1\right)\left[\left(\frac{y+1}{y-1}\right)^2 - 2\left(\frac{y+1}{y-1}\right)\cos\frac{2\pi}{5} + 1\right]\left[\left(\frac{y+1}{y-1}\right)^2 - 2\left(\frac{y+1}{y-1}\right)\cos\frac{4\pi}{5} + 1\right]$$

Multiply both sides by $(y - 1)^5$

$(y + 1)^5 - (y - 1)^5$

$$= (y + 1 - y + 1)\left[(y+1)^2 - 2(y+1)(y-1)\cos\frac{2\pi}{5} + (y-1)^2\right]\left[(y+1)^2 - 2(y+1)(y-1)\cos\frac{4\pi}{5} + (y-1)^2\right]$$

$$= 2\left[(y+1)^2 + (y-1)^2 - 2(y^2-1)\cos\frac{2\pi}{5}\right]\left[(y+1)^2 + (y-1)^2 - 2(y^2-1)\cos\frac{4\pi}{5}\right]$$

$$= 2\left[2y^2 + 2 - 2(y^2-1)\cos\frac{2\pi}{5}\right]\left[2y^2 + 2 - 2(y^2-1)\cos\frac{4\pi}{5}\right]$$

$$=8\left[\left(1-\cos\frac{2\pi}{5}\right)y^2+1+\cos\frac{2\pi}{5}\right]\left[\left(1-\cos\frac{4\pi}{5}\right)y^2+1+\cos\frac{4\pi}{5}\right]$$

$$=8\left[\left(1-1+2\sin^2\frac{\pi}{5}\right)y^2+1+2\cos^2\frac{\pi}{5}-1\right]\left[\left(1-1-2\sin^2\frac{2\pi}{5}\right)y^2+1+2\cos^2\frac{2\pi}{5}-1\right]$$

$$=8\left(2y^2\sin^2\frac{\pi}{5}+2\cos^2\frac{\pi}{5}\right)\left(2y^2\sin^2\frac{2\pi}{5}+2\cos^2\frac{2\pi}{5}\right)$$

$$=32\sin^2\frac{\pi}{5}\cdot\sin^2\frac{2\pi}{5}\left(y^2+\cot^2\frac{\pi}{5}\right)\left(y^2+\cot^2\frac{2\pi}{5}\right)$$

$$=32\cdot\frac{5}{16}\cdot\left(y^2+\cot^2\frac{\pi}{5}\right)\left(y^2+\cot^2\frac{2\pi}{5}\right)\quad\text{by the result of (**)}$$

$$\therefore\ (y+1)^5-(y-1)^5=10\left(y^2+\cot^2\frac{\pi}{5}\right)\left(y^2+\cot^2\frac{2\pi}{5}\right)$$

(c) (i) $10\left(y^2+\cot^2\frac{\pi}{5}\right)\left(y^2+\cot^2\frac{2\pi}{5}\right)=(y^5+5y^4+10y^3+10y^2+5y+1)-(y^5-5y^4+10y^3-10y^2+5y-1)$

$$10\left(y^2+\cot^2\frac{\pi}{5}\right)\left(y^2+\cot^2\frac{2\pi}{5}\right)=10y^4+20y^2+2$$

Compare the coefficient of $y^2$: $10\left(\cot^2\frac{\pi}{5}+\cot^2\frac{2\pi}{5}\right)=20$

$$\cot^2\frac{\pi}{5}+\cot^2\frac{2\pi}{5}=2\ \cdots\cdots\ (1)$$

$$1+\cot^2\frac{\pi}{5}+1+\cot^2\frac{2\pi}{5}=4$$

$$\csc^2\frac{\pi}{5}+\csc^2\frac{2\pi}{5}=4$$

(ii) Compare the constant term: $10\cot^2\frac{\pi}{5}\cot^2\frac{2\pi}{5}=2$

$$\cot\frac{\pi}{5}\cot\frac{2\pi}{5}=\frac{1}{\sqrt{5}}\quad\left(\because\ 0<\frac{\pi}{5},\frac{2\pi}{5}<\frac{\pi}{2}\quad\therefore\ 0<\cot\frac{\pi}{5},\cot\frac{2\pi}{5}\right)$$

$$\tan\frac{\pi}{5}\tan\frac{2\pi}{5}=\sqrt{5}\quad\cdots\cdots\ (2)$$

By (1): $\frac{1}{\tan^2\frac{\pi}{5}}+\frac{1}{\tan^2\frac{2\pi}{5}}=2$

$$\tan^2\frac{\pi}{5}+\tan^2\frac{2\pi}{5}=2\tan^2\frac{\pi}{5}\tan^2\frac{2\pi}{5}$$

$$\tan^2\frac{\pi}{5}+\tan^2\frac{2\pi}{5}=2\left(\sqrt{5}\right)^2\quad\text{by the result of (2)}$$

$$\tan^2\frac{\pi}{5}+\tan^2\frac{2\pi}{5}=10\ \cdots\cdots\ (3)$$

$$\left(\tan\frac{\pi}{5}-\tan\frac{2\pi}{5}\right)^2=\tan^2\frac{\pi}{5}+\tan^2\frac{2\pi}{5}-2\tan\frac{\pi}{5}\tan\frac{2\pi}{5}$$

$$\left(\tan\frac{\pi}{5}-\tan\frac{2\pi}{5}\right)^2=10-2\sqrt{5}\quad\text{by (3) and (2)}$$

$$\tan\frac{\pi}{5}-\tan\frac{2\pi}{5}=-\sqrt{10-2\sqrt{5}}\quad\left(\because\ \tan\frac{\pi}{5}<\tan\frac{2\pi}{5}\right)$$

## Generalisation

(a) $z^{2m+1} - 1 = 0 \Rightarrow z = 1, \text{cis}\,\frac{2k\pi}{2m+1}$, $k = -m, -m+1, \cdots, -1, 1, \cdots, m-1$ (where $m \in \mathbf{N}$)

(b) $z^{2m+1} - 1 = \prod_{k=-m}^{m}\left(z - cis\frac{2k\pi}{2m+1}\right)$

$$= (z-1)\prod_{k=-m}^{-1}\left(z - cis\frac{2k\pi}{2m+1}\right)\cdot\prod_{k=1}^{m}\left(z - cis\frac{2k\pi}{2m+1}\right)$$

$$= (z-1)\cdot\prod_{k=1}^{m}\left[\left(z - \cos\frac{2k\pi}{2m+1} - i\sin\frac{2k\pi}{2m+1}\right)\left(z - \cos\frac{2k\pi}{2m+1} + i\sin\frac{2k\pi}{2m+1}\right)\right]$$

$$= (z-1)\cdot\prod_{k=1}^{m}\left(z^2 - 2z\cos\frac{2k\pi}{2m+1} + 1\right) \quad \cdots\cdots (*)$$

$$\prod_{k=1}^{m}\left(z^2 - 2z\cos\frac{2k\pi}{2m+1} + 1\right) = \frac{z^{2m+1}-1}{z-1} = z^{2m} + z^{2m-1} + \cdots + 1$$

As $z \to 1$: $\prod_{k=1}^{m}\left(2 - 2\cos\frac{2k\pi}{2m+1}\right) = 2m + 1$

$$2^m \cdot \prod_{k=1}^{m}\left(1 - \cos\frac{2k\pi}{2m+1}\right) = 2m + 1$$

$$\prod_{k=1}^{m}\left(1 - 1 + 2\sin^2\frac{k\pi}{2m+1}\right) = \frac{2m+1}{2^m}$$

$$2^m \cdot \prod_{k=1}^{m}\left(\sin^2\frac{k\pi}{2m+1}\right) = \frac{2m+1}{2^m}$$

$$\left[\prod_{k=1}^{m}\left(\sin\frac{k\pi}{2m+1}\right)\right]^2 = \frac{2m+1}{2^{2m}} \quad \cdots\cdots (**)$$

Put $z = \frac{y+1}{y-1}$ into (*)

$$\left(\frac{y+1}{y-1}\right)^{2m+1} - 1 = \left(\frac{y+1}{y-1} - 1\right)\prod_{k=1}^{m}\left[\left(\frac{y+1}{y-1}\right)^2 - 2\cdot\frac{y+1}{y-1}\cdot\cos\frac{2k\pi}{2m+1} + 1\right]$$

Multiply both sides by $(y-1)^{2m+1}$

$$(y+1)^{2m+1} - (y-1)^{2m+1} = (y+1-y+1)\prod_{k=1}^{m}\left[(y+1)^2 - 2\cdot(y^2-1)\cdot\cos\frac{2k\pi}{2m+1} + (y-1)^2\right]$$

$$= 2\cdot\prod_{k=1}^{m}\left[2y^2 + 2 - 2\cdot(y^2-1)\cdot\cos\frac{2k\pi}{2m+1}\right]$$

$$= 2^{m+1}\cdot\prod_{k=1}^{m}\left[y^2 + 1 - (y^2-1)\cdot\cos\frac{2k\pi}{2m+1}\right]$$

$$= 2^{m+1}\cdot\prod_{k=1}^{m}\left[\left(1 - \cos\frac{2k\pi}{2m+1}\right)y^2 + 1 + \cos\frac{2k\pi}{2m+1}\right]$$

$$= 2^{m+1}\cdot\prod_{k=1}^{m}\left[\left(1 - 1 + 2\sin^2\frac{k\pi}{2m+1}\right)y^2 + 1 + 2\cos^2\frac{k\pi}{2m+1} - 1\right]$$

$$= 2^{2m+1}\cdot\prod_{k=1}^{m}\left(y^2\sin^2\frac{k\pi}{2m+1} + \cos^2\frac{k\pi}{2m+1}\right)$$

$$= 2^{2m+1}\cdot\prod_{k=1}^{m}\left[\sin^2\frac{k\pi}{2m+1}\left(y^2 + \cot^2\frac{k\pi}{2m+1}\right)\right]$$

$$=2^{2m+1}\cdot\prod_{k=1}^{m}\left(\sin^2\frac{k\pi}{2m+1}\right)\cdot\prod_{k=1}^{m}\left(y^2+\cot^2\frac{k\pi}{2m+1}\right)$$

$$=2^{2m+1}\cdot\frac{2m+1}{2^{2m}}\cdot\prod_{k=1}^{m}\left(y^2+\cot^2\frac{k\pi}{2m+1}\right)\quad\text{by the result of (**)}$$

$$\therefore\ (y+1)^{2m+1}-(y-1)^{2m+1}=(4m+2)\cdot\prod_{k=1}^{m}\left(y^2+\cot^2\frac{k\pi}{2m+1}\right)$$

(c) (i) $$(4m+2)\cdot\prod_{k=1}^{m}\left(y^2+\cot^2\frac{k\pi}{2m+1}\right)=\sum_{k=0}^{2m+1}C_k^{2m+1}y^{2m+1-k}-\sum_{k=0}^{2m+1}(-1)^kC_k^{2m+1}y^{2m+1-k}$$

$$(4m+2)\cdot\prod_{k=1}^{m}\left(y^2+\cot^2\frac{k\pi}{2m+1}\right)=2\cdot\sum_{k=0}^{m}C_{2k+1}^{2m+1}y^{2m-2k}$$

$$(2m+1)\cdot\prod_{k=1}^{m}\left(y^2+\cot^2\frac{k\pi}{2m+1}\right)=\sum_{k=0}^{m}C_{2k+1}^{2m+1}y^{2(m-k)}=C_1^{2m+1}y^{2m}+C_3^{2m+1}y^{2m-2}+\cdots+C_{2m+1}^{2m+1}$$

Compare coefficient of $y^{2m-2}$: $(2m+1)\cdot\sum_{k=1}^{m}\cot^2\frac{k\pi}{2m+1}=C_3^{2m+1}$

$$\sum_{k=1}^{m}\cot^2\frac{k\pi}{2m+1}=\frac{1}{2m+1}\cdot\frac{2m+1}{1}\times\frac{2m}{2}\times\frac{2m-1}{3}=\frac{m(2m-1)}{3}\quad\cdots\cdots(1)$$

$$\sum_{k=1}^{m}\left(1+\cot^2\frac{k\pi}{2m+1}\right)=m+\frac{m(2m-1)}{3}=\frac{2m(m+1)}{3}$$

$$\sum_{k=1}^{m}\csc^2\frac{k\pi}{2m+1}=\frac{2m(m+1)}{3}\quad\cdots\cdots(2)$$

(ii) By (c)(i), $(2m+1)\cdot\prod_{k=1}^{m}\left(y^2+\cot^2\frac{k\pi}{2m+1}\right)=\sum_{k=0}^{m}C_{2k+1}^{2m+1}y^{2(m-k)}$

Compare the constant term, $(2m+1)\cdot\prod_{k=1}^{m}\cot^2\frac{k\pi}{2m+1}=C_{2m+1}^{2m+1}$

$$\left(\prod_{k=1}^{m}\cot\frac{k\pi}{2m+1}\right)^2=\frac{1}{2m+1}$$

$$\prod_{k=1}^{m}\tan\frac{k\pi}{2m+1}=\sqrt{2m+1}\quad\cdots\cdots(3)\quad\left(\because\ 0<\frac{k\pi}{2m+1}<\frac{\pi}{2}\quad\therefore\ 0<\tan\frac{k\pi}{2m+1}\right)$$

By (**), $\left[\prod_{k=1}^{m}\left(\sin\frac{k\pi}{2m+1}\right)\right]^2=\frac{2m+1}{2^{2m}}$

$$\prod_{k=1}^{m}\sin\frac{k\pi}{2m+1}=\frac{\sqrt{2m+1}}{2^m}\quad\cdots\cdots(4)$$

(4)÷(3): $\prod_{k=1}^{m}\cos\frac{k\pi}{2m+1}=\frac{1}{2^m}\quad\cdots\cdots(5)$

$$\log(2m+1)+\sum_{k=1}^{m}\log\left(y^2+\cot^2\frac{k\pi}{2m+1}\right)=\log\left[\sum_{p=0}^{m}C_{2p+1}^{2m+1}y^{2(m-p)}\right]$$

Differentiate w.r.t. $y$:

$$\sum_{k=1}^{m}\frac{2y}{y^2+\cot^2\frac{k\pi}{2m+1}}=\frac{\sum_{k=0}^{m-1}(2m-2k)C_{2k+1}^{2m+1}y^{2m-2k-1}}{\sum_{k=0}^{m}C_{2p+1}^{2m+1}y^{2(m-p)}}=\frac{2mC_1^{2m+1}y^{2m-1}+(2m-2)C_3^{2m+1}y^{2m-3}+\cdots+2C_{2m-1}^{2m+1}\cdot y}{C_1^{2m+1}y^{2m}+C_3^{2m+1}y^{2m-2}+\cdots+C_{2m+1}^{2m+1}}$$

$$\sum_{k=1}^{m}\frac{2y\tan^2\frac{k\pi}{2m+1}}{1-\left(-y^2\tan^2\frac{k\pi}{2m+1}\right)}=2\sum_{k=1}^{m}\left[y\tan^2\frac{k\pi}{2m+1}\cdot\sum_{j=0}^{\infty}\left(-y^2\tan^2\frac{k\pi}{2m+1}\right)^j\right]$$

$$=2\sum_{k=1}^{m}\left[\sum_{j=0}^{\infty}(-1)^j\cdot\tan^{2j+2}\frac{k\pi}{2m+1}\cdot y^{2j+1}\right]$$

$$=2\sum_{j=0}^{\infty}(-1)^j\cdot\left(\sum_{k=1}^{m}\tan^{2j+2}\frac{k\pi}{2m+1}\right)\cdot y^{2j+1}$$

$$\text{RHS}=\frac{\sum_{k=0}^{m-1}(2m-2k)C_{2k+1}^{2m+1}y^{2m-2k-1}}{C_1^{2m+1}y^{2m}+C_3^{2m+1}y^{2m-2}+\cdots+C_{2m-1}^{2m+1}y^2+C_{2m+1}^{2m+1}}$$

$$=\frac{2\sum_{k=0}^{m-1}(m-k)C_{2k+1}^{2m+1}y^{2m-2k-1}}{1-\left(-\sum_{p=0}^{m-1}C_{2p+1}^{2m+1}y^{2(m-p)}\right)}$$

$$=2\sum_{k=0}^{m-1}(m-k)C_{2k+1}^{2m+1}y^{2m-2k-1}\cdot\sum_{q=0}^{\infty}\left[-\sum_{p=0}^{m-1}C_{2p+1}^{2m+1}y^{2(m-p)}\right]^q$$

$$\sum_{j=0}^{\infty}(-1)^j\cdot\left(\sum_{k=1}^{m}\tan^{2j+2}\frac{k\pi}{2m+1}\right)\cdot y^{2j+1}=\sum_{k=0}^{m-1}(m-k)C_{2k+1}^{2m+1}y^{2m-2k-1}\cdot\sum_{q=0}^{\infty}\left[-\sum_{p=0}^{m-1}C_{2p+1}^{2m+1}y^{2(m-p)}\right]^q$$

Compare coefficients of $y$: $\sum_{k=1}^{m}\tan^2\frac{k\pi}{2m+1}=C_{2m-1}^{2m+1}=m(2m+1)$ …… (6)

Compare coefficients of $y^3$:

$$-\sum_{k=1}^{m}\tan^4\frac{k\pi}{2m+1}=-\left(C_{2m-1}^{2m+1}\right)^2+2C_{2m-3}^{2m+1}$$

$$=-m^2(2m+1)^2+2\cdot\frac{(2m+1)(2m)(2m-1)(2m-2)}{1\cdot2\cdot3\cdot4}$$

$$=-m^2(2m+1)^2+\frac{1}{3}m(2m+1)(m-1)(2m-1)$$

$$=\frac{1}{3}m(2m+1)\left(2m^2-3m+1-6m^2-3m\right)$$

$$=-\frac{1}{3}m(2m+1)\left(4m^2+6m-1\right)$$

$$\sum_{k=1}^{m}\tan^4\frac{k\pi}{2m+1}=\frac{1}{3}m(2m+1)\left(4m^2+6m-1\right)\quad\cdots\cdots(7)$$

(d) $\tan\theta>\theta>\sin\theta\Rightarrow\cot^2\theta<\frac{1}{\theta^2}<\csc^2\theta$

$$\cot^2\frac{k\pi}{2m+1}<\left(\frac{2m+1}{k\pi}\right)^2<\csc^2\frac{k\pi}{2m+1}$$

$$\sum_{k=1}^{n}\cot^2\frac{k\pi}{2m+1}<\sum_{k=1}^{n}\left(\frac{2m+1}{k\pi}\right)^2<\sum_{k=1}^{n}\csc^2\frac{k\pi}{2m+1}$$

$\frac{m(2m-1)}{3}<\frac{(2m+1)^2}{\pi^2}\sum_{k=1}^{m}\frac{1}{k^2}<\frac{2m(m+1)}{3}$ by the result of (1) and (2).

$$\frac{\pi^2}{6}\left(\frac{2m}{2m+1}\right)\left(\frac{2m-1}{2m+1}\right)<\sum_{k=1}^{m}\frac{1}{k^2}<\frac{\pi^2}{6}\left(\frac{2m}{2m+1}\right)\left(\frac{2m+2}{2m+1}\right)$$

$$\frac{\pi^2}{6}\lim_{m\to\infty}\left(\frac{2m}{2m+1}\right)\left(\frac{2m-1}{2m+1}\right)\le\lim_{m\to\infty}\sum_{k=1}^{m}\frac{1}{k^2}\le\frac{\pi^2}{6}\lim_{m\to\infty}\left(\frac{2m}{2m+1}\right)\left(\frac{2m+2}{2m+1}\right)$$

$$\frac{\pi^2}{6}=\frac{\pi^2}{6}\lim_{m\to\infty}\left(\frac{1}{1+\frac{1}{2m}}\right)\left(\frac{1-\frac{1}{2m}}{1+\frac{1}{2m}}\right)\le\lim_{m\to\infty}\sum_{k=1}^{m}\frac{1}{k^2}\le\frac{\pi^2}{6}\lim_{m\to\infty}\left(\frac{1}{1+\frac{1}{2m}}\right)\left(\frac{1+\frac{2}{2m}}{1+\frac{1}{2m}}\right)=\frac{\pi^2}{6}$$

By squeezing principle, $\lim_{m\to\infty}\sum_{k=1}^{m}\frac{1}{k^2}=\frac{\pi^2}{6}$ …… (8)

By (c)(i), $(2m+1)\cdot\prod_{k=1}^{m}\left(y^2+\cot^2\frac{k\pi}{2m+1}\right)=\sum_{k=0}^{m}C_{2k+1}^{2m+1}y^{2(m-k)}$

Compare coefficient of $y^{2(m-2)}$: $(2m+1)\cdot\sum_{p<q}\cot^2\frac{p\pi}{2m+1}\cot^2\frac{q\pi}{2m+1}=C_5^{2m+1}$

$$\sum_{p<q}\cot^2\frac{p\pi}{2m+1}\cot^2\frac{q\pi}{2m+1}=\frac{1}{2m+1}\cdot\frac{(2m+1)(2m)(2m-1)(2m-2)(2m-3)}{1\cdot2\cdot3\cdot4\cdot5}$$

$$\sum_{p<q}\cot^2\frac{p\pi}{2m+1}\cot^2\frac{q\pi}{2m+1}=\frac{1}{30}\cdot(m-1)m(2m-3)(2m-1)\quad\cdots\cdots(9)$$

$$\left(\sum_{k=1}^{m}\cot^2\frac{k\pi}{2m+1}\right)^2=\sum_{k=1}^{m}\cot^4\frac{k\pi}{2m+1}+2\sum_{p<q}\cot^2\frac{p\pi}{2m+1}\cot^2\frac{q\pi}{2m+1}$$

$$\left[\frac{m(2m-1)}{3}\right]^2=\sum_{k=1}^{m}\cot^4\frac{k\pi}{2m+1}+\frac{1}{15}\cdot(m-1)m(2m-3)(2m-1)\quad\text{by (1) and (9)}$$

$$\begin{aligned}\sum_{k=1}^{m}\cot^4\frac{k\pi}{2m+1}&=\frac{m^2(2m-1)^2}{9}-\frac{1}{15}\cdot(m-1)m(2m-3)(2m-1)\\&=\frac{1}{45}\cdot m(2m-1)[5m(2m-1)-3(m-1)(2m-3)]\\&=\frac{1}{45}\cdot m(2m-1)(10m^2-5m-6m^2+15m-9)\end{aligned}$$

$$\sum_{k=1}^{m}\cot^4\frac{k\pi}{2m+1}=\frac{1}{45}\cdot m(2m-1)(4m^2+10m-9)\quad\cdots\cdots(10)$$

$$\sum_{k=1}^{m}\left(\csc^2\frac{k\pi}{2m+1}-1\right)^2=\frac{1}{45}\cdot m(2m-1)(4m^2+10m-9)$$

$$\sum_{k=1}^{m}\csc^4\frac{k\pi}{2m+1}-2\sum_{k=1}^{m}\csc^2\frac{k\pi}{2m+1}+m=\frac{1}{45}\cdot m(2m-1)(4m^2+10m-9)$$

$$\begin{aligned}\sum_{k=1}^{m}\csc^4\frac{k\pi}{2m+1}&=2\sum_{k=1}^{m}\csc^2\frac{k\pi}{2m+1}+\frac{1}{45}\cdot m(2m-1)(4m^2+10m-9)-m\\&=2\cdot\frac{2m(m+1)}{3}+\frac{1}{45}\cdot m(2m-1)(4m^2+10m-9)-m\quad\text{by (2)}\\&=\frac{m}{45}\cdot(60m+60+8m^3+16m^2-28m+9-45)\\&=\frac{m}{45}\cdot(8m^3+16m^2+32m+24)\end{aligned}$$

$$\sum_{k=1}^{m}\csc^4\frac{k\pi}{2m+1}=\frac{8m}{45}\cdot(m^3+2m^2+4m+3)=\frac{8m}{45}\cdot(m+1)(m^2+m+3)\quad\cdots\cdots(11)$$

Similarly, $\tan\theta > \theta > \sin\theta \Rightarrow \cot^4\theta < \frac{1}{\theta^4} < \csc^4\theta$

$$\cot^4\frac{k\pi}{2m+1} < \left(\frac{2m+1}{k\pi}\right)^4 < \csc^4\frac{k\pi}{2m+1}$$

$$\sum_{k=1}^{n}\cot^4\frac{k\pi}{2m+1} < \sum_{k=1}^{n}\left(\frac{2m+1}{k\pi}\right)^4 < \sum_{k=1}^{n}\csc^4\frac{k\pi}{2m+1}$$

$\frac{1}{45}\cdot m(2m-1)(4m^2+10m-9) < \frac{(2m+1)^4}{\pi^4}\sum_{k=1}^{m}\frac{1}{k^4} < \frac{8m}{45}\cdot(m+1)(m^2+m+3)$ by (10) and (11).

$$\frac{\pi^4}{90}\left(\frac{2m}{2m+1}\right)\left(\frac{2m-1}{2m+1}\right)\left[\frac{4m^2+10m-9}{(2m+1)^2}\right] < \sum_{k=1}^{m}\frac{1}{k^4} < \frac{\pi^4}{90}\left(\frac{2m}{2m+1}\right)\left(\frac{2m+2}{2m+1}\right)\left[\frac{4m^2+4m+12}{(2m+1)^2}\right]$$

Take limit $m\to\infty$, $\lim_{m\to\infty}\sum_{k=1}^{m}\frac{1}{k^4} = \frac{\pi^4}{90}$ …… (12)

**Variation**

(a) $z^{2m}-1=0 \Rightarrow z=1,-1,\ \text{cis}\,\dfrac{2k\pi}{2m}$ (i.e. $\text{cis}\,\dfrac{k\pi}{m}$), $k=-m+1,\cdots,-1,1,\cdots,m-1$ (where $m\in\mathbf{N}$)

(b) $$z^{2m}-1=(z+1)(z-1)\prod_{k=1}^{m-1}\left(z-\cos\frac{k\pi}{m}+i\sin\frac{k\pi}{m}\right)\left(z-\cos\frac{k\pi}{m}-i\sin\frac{k\pi}{m}\right)$$

$$=(z^2-1)\cdot\prod_{k=1}^{m-1}\left(z^2-2z\cos\frac{k\pi}{m}+1\right)\quad\cdots\cdots(*)$$

$$\prod_{k=1}^{m-1}\left(z^2-2z\cos\frac{k\pi}{m}+1\right)=\frac{z^{2m}-1}{z^2-1}=z^{2m-2}+z^{2m-4}+\cdots+1$$

As $z\to 1$: $\displaystyle\prod_{k=1}^{m-1}\left(2-2\cos\frac{k\pi}{m}\right)=m$

$$2^{m-1}\cdot\prod_{k=1}^{m-1}\left(1-\cos\frac{k\pi}{m}\right)=m$$

$$\prod_{k=1}^{m-1}\left(1-1+2\sin^2\frac{k\pi}{2m}\right)=\frac{m}{2^{m-1}}$$

$$2^{m-1}\cdot\prod_{k=1}^{m-1}\left(\sin^2\frac{k\pi}{2m}\right)=\frac{m}{2^{m-1}}$$

$$\left[\prod_{k=1}^{m-1}\left(\sin\frac{k\pi}{2m}\right)\right]^2=\frac{m}{2^{2(m-1)}}\quad\cdots\cdots(**)$$

Put $z=\dfrac{y+1}{y-1}$ into (*)

$$\left(\frac{y+1}{y-1}\right)^{2m}-1=\left[\left(\frac{y+1}{y-1}\right)^2-1\right]\prod_{k=1}^{m-1}\left[\left(\frac{y+1}{y-1}\right)^2-2\cdot\frac{y+1}{y-1}\cdot\cos\frac{k\pi}{m}+1\right]$$

Multiply both sides by $(y-1)^{2m}$

$$(y+1)^{2m}-(y-1)^{2m}=(y^2+2y+1-y^2+2y-1)\prod_{k=1}^{m-1}\left[(y+1)^2-2\cdot(y^2-1)\cdot\cos\frac{k\pi}{m}+(y-1)^2\right]$$

$$=4y\cdot\prod_{k=1}^{m-1}\left[2y^2+2-2\cdot(y^2-1)\cdot\cos\frac{k\pi}{m}\right]$$

$$=2^{m+1}y\cdot\prod_{k=1}^{m-1}\left[y^2+1-(y^2-1)\cdot\cos\frac{k\pi}{m}\right]$$

$$=2^{m+1}y\cdot\prod_{k=1}^{m-1}\left[\left(1-\cos\frac{k\pi}{m}\right)y^2+1+\cos\frac{k\pi}{m}\right]$$

$$=2^{m+1}y\cdot\prod_{k=1}^{m-1}\left[\left(1-1+2\sin^2\frac{k\pi}{2m}\right)y^2+1+2\cos^2\frac{k\pi}{2m}-1\right]$$

$$=2^{2m}y\cdot\prod_{k=1}^{m-1}\left(y^2\sin^2\frac{k\pi}{2m}+\cos^2\frac{k\pi}{2m}\right)=2^{2m}y\cdot\prod_{k=1}^{m-1}\left[\sin^2\frac{k\pi}{2m}\left(y^2+\cot^2\frac{k\pi}{2m}\right)\right]$$

$$=2^{2m}y\cdot\prod_{k=1}^{m-1}\left(\sin^2\frac{k\pi}{2m}\right)\cdot\prod_{k=1}^{m-1}\left(y^2+\cot^2\frac{k\pi}{2m}\right)$$

$$=2^{2m}y\cdot\frac{m}{2^{2(m-1)}}\cdot\prod_{k=1}^{m-1}\left(y^2+\cot^2\frac{k\pi}{2m}\right)\quad\text{by the result of }(**)$$

$$\therefore\ (y+1)^{2m}-(y-1)^{2m}=4my\cdot\prod_{k=1}^{m-1}\left(y^2+\cot^2\frac{k\pi}{2m}\right)$$

(c) (i) $4my\cdot\prod_{k=1}^{m-1}\left(y^2+\cot^2\frac{k\pi}{2m}\right)=\sum_{k=0}^{2m}C_k^{2m}y^{2m-k}-\sum_{k=0}^{2m}(-1)^k C_k^{2m}y^{2m-k}$

$$4my\cdot\prod_{k=1}^{m-1}\left(y^2+\cot^2\frac{k\pi}{2m}\right)=2\cdot\sum_{k=0}^{m-1}C_{2k+1}^{2m}y^{2m-2k-1}$$

$$2m\prod_{k=1}^{m-1}\left(y^2+\cot^2\frac{k\pi}{2m}\right)=\sum_{k=0}^{m-1}C_{2k+1}^{2m}y^{2m-2k-2}=C_1^{2m}y^{2m-2}+C_3^{2m}y^{2m-4}+\cdots+C_{2m-1}^{2m}$$

Compare coefficient of $y^{2m-4}$: $2m\cdot\sum_{k=1}^{m-1}\cot^2\frac{k\pi}{2m}=C_3^{2m}$

$$\sum_{k=1}^{m-1}\cot^2\frac{k\pi}{2m}=\frac{1}{2m}\cdot\frac{2m}{1}\times\frac{2m-1}{2}\times\frac{2m-2}{3}=\frac{(m-1)(2m-1)}{3}\quad\ldots\ldots(1)$$

$$\sum_{k=1}^{m-1}\left(1+\cot^2\frac{k\pi}{2m}\right)=m-1+\frac{(m-1)(2m-1)}{3}=\frac{2(m-1)(m+1)}{3}$$

$$\sum_{k=1}^{m-1}\csc^2\frac{k\pi}{2m}=\frac{2(m-1)(m+1)}{3}\quad\ldots\ldots(2)$$

(ii) By (c)(i), $2m\prod_{k=1}^{m-1}\left(y^2+\cot^2\frac{k\pi}{2m}\right)=\sum_{k=0}^{m-1}C_{2k+1}^{2m}y^{2m-2k-2}$

Compare the constant term, $2m\cdot\prod_{k=1}^{m-1}\cot^2\frac{k\pi}{2m}=C_{2m-1}^{2m}$

$$\left(\prod_{k=1}^{m-1}\cot\frac{k\pi}{2m}\right)^2=1$$

$$\prod_{k=1}^{m-1}\tan\frac{k\pi}{2m}=1\quad\cdots\cdots(3)\quad\left(\because\ 0<\frac{k\pi}{2m}<\frac{\pi}{2}\quad\therefore\ 0<\tan\frac{k\pi}{2m}\right)$$

By (**), $\left[\prod_{k=1}^{m-1}\left(\sin\frac{k\pi}{2m}\right)\right]^2=\frac{m}{2^{2(m-1)}}$

$$\prod_{k=1}^{m-1}\sin\frac{k\pi}{2m}=\frac{\sqrt{m}}{2^{m-1}}\quad\cdots\cdots(4)$$

(4)÷(3): $\prod_{k=1}^{m-1}\cos\frac{k\pi}{2m}=\frac{\sqrt{m}}{2^{m-1}}\quad\cdots\cdots(5)$

$$\log 2m+\sum_{k=1}^{m-1}\log\left(y^2+\cot^2\frac{k\pi}{2m}\right)=\log\left(\sum_{k=0}^{m-1}C_{2k+1}^{2m}y^{2m-2k-2}\right)$$

Differentiate w.r.t. $y$:

$$\sum_{k=1}^{m-1}\frac{2y}{y^2+\cot^2\frac{k\pi}{2m}}=\frac{\sum_{k=0}^{m-2}(2m-2k-2)C_{2k+1}^{2m}y^{2m-2k-3}}{\sum_{p=0}^{m-1}C_{2p+1}^{2m}y^{2m-2p-2}}$$

$$\sum_{k=1}^{m-1}\frac{2y}{y^2+\cot^2\frac{k\pi}{2m}}=\frac{(2m-2)C_1^{2m}y^{2m-3}+(2m-4)C_3^{2m}y^{2m-5}+\cdots+2C_{2m-3}^{2m}\cdot y}{C_1^{2m}y^{2m-2}+C_3^{2m}y^{2m-4}+\cdots+C_{2m-1}^{2m}}$$

L.H.S. $=\sum_{k=1}^{m-1}\frac{2y}{y^2+\cot^2\frac{k\pi}{2m}}=\sum_{k=1}^{m-1}\frac{2y\tan^2\frac{k\pi}{2m}}{1-\left(-y^2\tan^2\frac{k\pi}{2m}\right)}=2\sum_{k=1}^{m-1}\left[y\tan^2\frac{k\pi}{2m}\cdot\sum_{j=0}^{\infty}\left(-y^2\tan^2\frac{k\pi}{2m}\right)^j\right]$

$$=2\sum_{k=1}^{m-1}\left[\sum_{j=0}^{\infty}(-1)^j\cdot\tan^{2j+2}\frac{k\pi}{2m}\cdot y^{2j+1}\right]=2\sum_{j=0}^{\infty}(-1)^j\cdot\left(\sum_{k=1}^{m-1}\tan^{2j+2}\frac{k\pi}{2m}\right)\cdot y^{2j+1}$$

$$\text{R.H.S.}=\frac{(2m-2)C_1^{2m}y^{2m-3}+(2m-4)C_3^{2m}y^{2m-5}+\cdots+2C_{2m-3}^{2m}\cdot y}{C_1^{2m}y^{2m-2}+C_3^{2m}y^{2m-4}+\cdots+C_{2m-1}^{2m}}$$

$$=\frac{2\sum_{k=0}^{m-2}(m-k-1)C_{2k+1}^{2m}y^{2m-2k-3}}{2m\left[1-\left(-\frac{1}{2m}\sum_{p=0}^{m-2}C_{2p+1}^{2m}y^{2m-2p-2}\right)\right]}$$

$$=\frac{1}{m}\sum_{k=0}^{m-2}(m-k-1)C_{2k+1}^{2m}y^{2m-2k-3}\cdot\sum_{q=0}^{\infty}\left[-\frac{1}{2m}\sum_{p=0}^{m-2}C_{2p+1}^{2m}y^{2m-2p-2}\right]^q$$

$$2\sum_{j=0}^{\infty}(-1)^j\cdot\left(\sum_{k=1}^{m-1}\tan^{2j+2}\frac{k\pi}{2m}\right)\cdot y^{2j+1}=\frac{1}{m}\sum_{k=0}^{m-2}(m-k-1)C_{2k+1}^{2m}y^{2m-2k-3}\cdot\sum_{q=0}^{\infty}\left[-\frac{1}{2m}\sum_{p=0}^{m-2}C_{2p+1}^{2m}y^{2m-2p-2}\right]^q$$

Compare coefficients of $y$: $2\sum_{k=1}^{m-1}\tan^2\frac{k\pi}{2m}=\frac{1}{m}C_{2m-3}^{2m}=\frac{1}{m}\cdot\frac{(2m)(2m-1)(2m-2)}{1\cdot 2\cdot 3}$

$$\sum_{k=1}^{m-1}\tan^2\frac{k\pi}{2m}=\frac{(m-1)(2m-1)}{3}\quad\cdots\cdots(6)$$

In particular, put $m = 90$.

$$\tan^2 1^\circ+\tan^2 2^\circ+\cdots+\tan^2 89^\circ=\frac{(90-1)(180-1)}{3}=\frac{15931}{3}=5310.333$$

Compare coefficients of $y^3$:

$$-2\sum_{k=1}^{m-1}\tan^4\frac{k\pi}{2m}=\frac{1}{m}\left[-\frac{1}{2m}\left(C_{2m-3}^{2m}\right)^2+2C_{2m-5}^{2m}\right]$$

$$\sum_{k=1}^{m-1}\tan^4\frac{k\pi}{2m}=\frac{1}{2m}\cdot\left\{\frac{1}{2m}\left[\frac{(2m)(2m-1)(2m-2)}{1\cdot 2\cdot 3}\right]^2-2\cdot\frac{(2m)(2m-1)(2m-2)(2m-3)(2m-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\right\}$$

$$=\frac{1}{2m}\cdot\left[\frac{(2m)(m-1)^2(2m-1)^2}{9}-\frac{(2m)(2m-1)(m-1)(2m-3)(m-2)}{15}\right]$$

$$=\frac{(m-1)(2m-1)}{45}\cdot[5(m-1)(2m-1)-3(2m-3)(m-2)]$$

$$=\frac{(m-1)(2m-1)}{45}\cdot\left[5\left(2m^2-3m+1\right)-3\left(2m^2-7m+6\right)\right]$$

$$\sum_{k=1}^{m}\tan^4\frac{k\pi}{2m}=\frac{(m-1)(2m-1)}{45}\cdot\left(4m^2+6m-13\right)\quad\cdots\cdots(7)$$