Hard Summation

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Hard summation using complex numbers:

(*a*) If *n* is a positive integer, show that

$$x^{2n} - 2a^n x^n \cos n\theta + a^{2n} = \prod_{k=0}^{n-1}\left[x^2 - 2ax\cos\left(\theta + \frac{2k\pi}{n}\right) + a^2\right]$$

(*b*) By taking $x = a = 1$ and suitable values of θ, show that

$$\prod_{k=0}^{n-1}\sin^2\left(\alpha + \frac{k\pi}{n}\right) + \prod_{k=0}^{n-1}\cos^2\left(\alpha + \frac{k\pi}{n}\right) = \begin{cases} 2^{2-2n} & \text{when } n \text{ is odd} \\ 2^{3-2n}\sin^2 n\alpha & \text{when } n \text{ is even}\end{cases}.$$

(*c*) Deduce that $\dfrac{nx^{n-1}\left(x^n - a^n \cos n\theta\right)}{x^{2n} - 2x^n a^n \cos n\theta + a^{2n}} = \displaystyle\sum_{k=0}^{n-1}\frac{x - a\cos\left(\theta + \frac{2k\pi}{n}\right)}{x^2 - 2xa\cos\left(\theta + \frac{2k\pi}{n}\right) + a^2}$ and

$$\sum_{k=0}^{n-1}\frac{1}{1 - \cos\left(\theta + \frac{2k\pi}{n}\right)} = \frac{n^2}{1 - \cos n\theta}$$

(*a*) Proved previously.

(*b*) $x = a = 1$, $\theta = 2\alpha$, $2 - 2\cos 2n\alpha = \displaystyle\prod_{k=0}^{n-1}\left[2 - 2\cos\left(2\alpha + \frac{2k\pi}{n}\right)\right]$

$$4\sin^2 n\alpha = \prod_{k=0}^{n-1}\left[4\sin^2\left(\alpha + \frac{k\pi}{n}\right)\right]$$

$$\prod_{k=0}^{n-1}\sin^2\left(\alpha + \frac{k\pi}{n}\right) = 2^{2-2n}\sin^2 n\alpha \quad \text{........(1)}$$

$x = a = 1$, $\theta = 2\alpha + \pi$, $2 - 2\cos(2n\alpha + n\pi) = \displaystyle\prod_{k=0}^{n-1}\left[2 - 2\cos\left(2\alpha + \frac{2k\pi}{n} + \pi\right)\right]$

$$4\sin^2\left(n\alpha + \frac{n\pi}{2}\right) = \prod_{k=0}^{n-1}\left[4\sin^2\left(\alpha + \frac{k\pi}{n} + \frac{\pi}{2}\right)\right]$$

$$\prod_{k=0}^{n-1}\left[\cos^2\left(\alpha + \frac{k\pi}{n}\right)\right] = \begin{cases} 2^{2-2n}\cos^2 n\alpha & \text{when } n \text{ is odd} \\ 2^{2-2n}\sin^2 n\alpha & \text{when } n \text{ is even}\end{cases} \quad \text{.........(2)}$$

(1) + (2) $\displaystyle\prod_{k=0}^{n-1}\sin^2\left(\alpha + \frac{k\pi}{n}\right) + \prod_{k=0}^{n-1}\cos^2\left(\alpha + \frac{k\pi}{n}\right) = \begin{cases} 2^{2-2n} & \text{when } n \text{ is odd} \\ 2^{3-2n}\sin^2 n\alpha & \text{when } n \text{ is even}\end{cases}$

(*c*) Using (*a*), $\ln[x^{2n} - 2a^n x^n \cos n\theta + a^{2n}] = \ln\displaystyle\prod_{k=0}^{n-1}\left[x^2 - 2ax\cos\left(\theta + \frac{2k\pi}{n}\right) + a^2\right] = \sum_{k=0}^{n-1}\ln\left[x^2 - 2ax\cos\left(\theta + \frac{2k\pi}{n}\right) + a^2\right]$

Differentiate both sides w.r.t. *x*. $\dfrac{nx^{n-1}\left(x^n - a^n \cos n\theta\right)}{x^{2n} - 2x^n a^n \cos n\theta + a^{2n}} = \displaystyle\sum_{k=0}^{n-1}\frac{x - a\cos\left(\theta + \frac{2k\pi}{n}\right)}{x^2 - 2xa\cos\left(\theta + \frac{2k\pi}{n}\right) + a^2}$

Consider the equation: $\left(\dfrac{x+i}{x-i}\right)^n = \operatorname{cis} n\theta$, where *n* is a positive integer.

$\left(\dfrac{x+i}{x-i}\right)^n = \operatorname{cis}(n\theta + 2k\pi)$, $k = 0, 1, 2, \ldots, n-1$

$\dfrac{x+i}{x-i} = \operatorname{cis}\left(\theta + \dfrac{2k\pi}{n}\right)$, $k = 0, 1, 2, \ldots, n-1$

Let $\omega = \text{cis}\left(\theta + \frac{2k\pi}{n}\right)$, $k = 0, 1, 2, \ldots, n - 1$

$x + i = \omega x - \omega i$, $k = 0, 1, 2, \ldots, n - 1$

$x(1 - \omega) = -i(1 + \omega)$, $k = 0, 1, 2, \ldots, n - 1$

$x = -i \cdot \frac{1+\omega}{1-\omega}$, $k = 0, 1, 2, \ldots, n - 1$

$$x = -i \cdot \frac{\omega^{\frac{1}{2}}\left(\omega^{\frac{1}{2}} + \omega^{-\frac{1}{2}}\right)}{-\omega^{\frac{1}{2}}\left(\omega^{\frac{1}{2}} - \omega^{-\frac{1}{2}}\right)}, \quad k = 0, 1, 2, \ldots, n - 1$$

$$x = i \cdot \frac{2\cos\left(\frac{\theta}{2} + \frac{k\pi}{n}\right)}{2i\sin\left(\frac{\theta}{2} + \frac{k\pi}{n}\right)}, \quad k = 0, 1, 2, \ldots, n - 1$$

$$x = \cot\left(\frac{\theta}{2} + \frac{k\pi}{n}\right), \quad k = 0, 1, 2, \ldots, n - 1$$

On the other hand, let $z = \text{cis}\, n\theta$, $\left(\frac{x+i}{x-i}\right)^n = \text{cis} n\theta$ is equivalent to $(x + i)^n = (x - i)^n z$

$$x^n + nix^{n-1} - \frac{n(n-1)}{2}x^{n-2} + \ldots\ldots = z[\, x^n - nix^{n-1} - \frac{n(n-1)}{2}x^{n-2} + \ldots\ldots]$$

$$(1 - z)x^n + (ni + niz)x^{n-1} + \frac{n(n-1)}{2}(z-1)x^{n-2} + \ldots\ldots = 0$$

Using the relation between the sum and product of roots,

$$\sum_{k=0}^{n-1}\cot\left(\frac{\theta}{2} + \frac{k\pi}{n}\right) = -\frac{ni(1+z)}{1-z}$$

$$= -\frac{niz^{\frac{1}{2}}\left(z^{\frac{1}{2}} + z^{-\frac{1}{2}}\right)}{-z^{\frac{1}{2}}\left(z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right)}$$

$$= \frac{ni\left(2\cos\frac{n\theta}{2}\right)}{2i\sin\frac{n\theta}{2}}$$

$$= n\cot\frac{n\theta}{2}$$

$$\sum_{j\neq m}\cot\left(\frac{\theta}{2}+\frac{j\pi}{n}\right)\cot\left(\frac{\theta}{2}+\frac{m\pi}{n}\right)=\frac{n(n-1)}{2}\cdot\frac{\cos n\theta-1+i\sin n\theta}{1-\cos n\theta-i\sin n\theta}=-\frac{n(n-1)}{2}$$

$$\sum_{k=0}^{n-1}\cot^2\left(\frac{\theta}{2}+\frac{k\pi}{n}\right)=\left[\sum_{k=0}^{n-1}\cot\left(\frac{\theta}{2}+\frac{k\pi}{n}\right)\right]^2-2\left[\sum_{j\neq m}\cot\left(\frac{\theta}{2}+\frac{j\pi}{n}\right)\cot\left(\frac{\theta}{2}+\frac{m\pi}{n}\right)\right]$$

$$=n^2\cot^2\frac{n\theta}{2}+n(n-1)$$

$$\sum_{k=0}^{n-1}\frac{1}{1-\cos\left(\theta+\frac{2k\pi}{n}\right)}=\sum_{k=0}^{n-1}\frac{1}{2\sin^2\left(\frac{\theta}{2}+\frac{k\pi}{n}\right)}$$

$$=\frac{1}{2}\sum_{k=0}^{n-1}\csc^2\left(\frac{\theta}{2}+\frac{k\pi}{n}\right)$$

$$=\frac{1}{2}\sum_{k=0}^{n-1}\left[\cot^2\left(\frac{\theta}{2}+\frac{k\pi}{n}\right)+1\right]$$

$$=\frac{n^2}{2}\cot^2\frac{n\theta}{2}+\frac{n(n-1)}{2}+\frac{n}{2}$$

$$=\frac{n^2}{2}\left(\cot^2\frac{n\theta}{2}+1\right)$$

$$=\frac{n^2}{2}\csc^2\frac{n\theta}{2}$$

$$=\frac{n^2}{2\sin^2\frac{n\theta}{2}}$$

$$=\frac{n^2}{1-\cos n\theta}$$

$$\sum_{k=0}^{n-1}\frac{1}{1-\cos\left(\theta+\frac{2k\pi}{n}\right)}=\frac{n^2}{1-\cos n\theta}$$