$\tan^2 1° + \tan^2 2° + \ldots + \tan^2 89°$

Created by Mr. Francis Hung Last updated: 21 April 2011

To find $\tan^2 1° + \tan^2 2° + \ldots + \tan^2 89°$.

Consider the equation $\left(\frac{1+ix}{1-ix}\right)^{180} = 1$, where $i = \sqrt{-1}$

$\frac{1+ix}{1-ix} = \cos\frac{2k\pi}{180} + i\sin\frac{2k\pi}{180}$, $k = 0, 1, 2, \ldots 179$

$\frac{1+ix}{1-ix} = \cos\frac{k\pi}{90} + i\sin\frac{k\pi}{90}$

Let $\omega = \cos\frac{k\pi}{90} + i\sin\frac{k\pi}{90}$

$\frac{1+ix}{1-ix} = \omega$

$1 + ix = \omega(1 - ix)$

$(1 + \omega)ix = \omega - 1$

$$x = \frac{\omega - 1}{i(\omega+1)} = \frac{\omega^{\frac{1}{2}} - \omega^{-\frac{1}{2}}}{i\left(\omega^{\frac{1}{2}} + \omega^{-\frac{1}{2}}\right)} = \frac{2i\sin\frac{k\pi}{180}}{2i\cos\frac{k\pi}{180}} = \tan\frac{k\pi}{180},\ k = 0, 1, 2, \ldots, 179$$

But the denominator $\neq 0$, $x \neq -i$

Also when $k = 90$, $x = \tan\frac{\pi}{2}$ which is undefined, therefore rejected.

From the equation: $\left(\frac{1+ix}{1-ix}\right)^{180} = 1 \Rightarrow (1 + ix)^{180} - (1 - ix)^{180} = 0$

Let $f(x) = (1 + ix)^{180} - (1 - ix)^{180}$, $f(x)$ can be expanded by binomial theorem

$f(x) = a_{180}x^{180} + a_{179}x^{179} + \ldots + a_1x + a_0$, where $a_r = {}_{180}C_r\,(i)^r[1 - (-1)^r]$, $0 \le r \le 180$

According to the formula, $a_r = 0$ when r is even, $a_r = {}_{180}C_r \times 2(i)^r$ when r is odd

$\therefore f(x) = a_{179}x^{179} + a_{177}x^{177} + \ldots + a_3x^3 + a_1x$

The roots of this equation are: $x = \tan\frac{k\pi}{180}$, $k = 0, 1, 2, \ldots, 89, 91, \ldots, 179$

Let $g(x) = \frac{f(x)}{x} = a_{179}x^{178} + a_{177}x^{176} + \cdots + a_3x^2 + a_1$,

The roots of $g(x)$ are $\tan\frac{k\pi}{180}$, $k = 1, 2, \ldots, 89, 91, \ldots, 179$

Consider $g(\sqrt{x}) = a_{179}x^{89} + a_{177}x^{88} + \cdots + a_3x + a_1 = 0$

The roots are $\tan^2 1°, \tan^2 2°, \ldots, \tan^2 89°$

$\tan^2 1° + \tan^2 2° + \ldots + \tan^2 89°$ = sum of roots of the equation of $g(\sqrt{x}) = 0$

$$\tan^2 1° + \tan^2 2° + \ldots + \tan^2 89° = -\frac{a_{177}}{a_{179}} = -\frac{{}_{180}C_{177} \times 2i^{177}}{{}_{180}C_{179} \times 2i^{179}} = \frac{15931}{3} = 5310.333$$