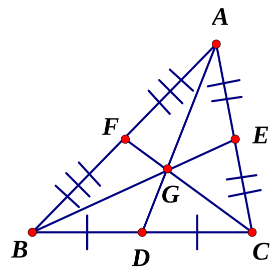
Centroid

Created by Mr. Francis Hung on 21 April 2011 Last updated: 2021-09-03

Theorem: The three **medians** of a triangle are **concurrent** at a point.



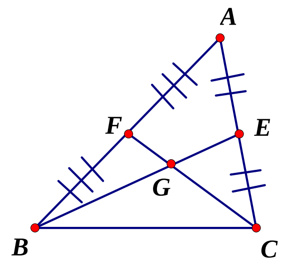
In ΔABC, let D, E and F be the mid points of

BC, CA and AB respectively.

The theorem says that the medians AD, BE

and CF meet at a point G.

G is called the **centroid** of the triangle.

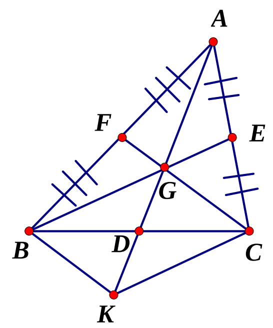


Proof: Suppose the medians BE and CF intersects at G.

To show that the line AG produced cuts BC at D is a median.

Join AG, and produce it to K, so that $AG = GK$.

Suppose AGK cuts BC at D. Join BK , CK .



$\because$ $AG = GK$ (by construction)

$AE = EC$ (E is the mid point)

By mid point theorem, $GE = \frac{1}{2}KC$, GE // KC

$\because$ $AG = GK$ (by construction)

$AF = FB$ (F is the mid point)

By mid point theorem, $GF = \frac{1}{2}KB$, GF // KB

$\because$ GE // KC and GF // KB

$\therefore$ $BKCG$ is a parallelogram.

$BD = DC$ (diagonals of a parallelogram.)

Therefore, the theorem is proved.

Furthermore, $GE = \frac{1}{2}KC = \frac{1}{2}BG$

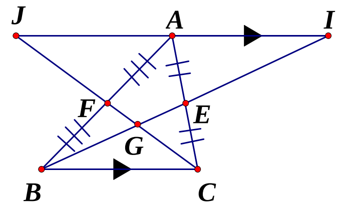
$$GF = \frac{1}{2}KB = \frac{1}{2}CG$$

so $BG : GE = CG : GF = 2 : 1$.

Similarly we can prove that $AG : GD = 2 : 1$.

The centriod divides each median in the ratio $2 : 1$.

Method 2 To prove that the 3 medians are **concurrent** at the **centroid**.
Suppose the medians BE and CF intersects at G.
Try to show that the line AG produced cuts BC at D is a median.
Draw a line JAI // BC, cutting BE produced at I, and CF produced at J.



$\angle IAE = \angle BCE$ (alt. ∠s, JI // BC)
$AE = EC$ (given that E is a mid-point)
$\angle AEI = \angle CEB$ (vert. opp. ∠s)
$\therefore \Delta AEI \cong \Delta CEB$ (A.S.A.)
$\mathbf{AI = BC}$ ⋯⋯ (1) (corr. sides, ≅ Δs)
$BE = EI$ ⋯⋯ (2) (corr. sides, ≅ Δs)
$\angle JAF = \angle CBF$ (alt. ∠s, JI // BC)
$AF = FB$ (given that F is a mid-point)
$\angle AFJ = \angle BFC$ (vert. opp. ∠s)
$\therefore \Delta AFJ \cong \Delta BFC$ (A.S.A.)
$\mathbf{JA = BC}$ ⋯⋯ (3) (corr. sides, ≅ Δs)
$CF = FJ$ ⋯⋯ (4) (corr. sides, ≅ Δs)
By (1) and (3), $JI = 2BC$ ⋯⋯ (5)
$\angle JIG = \angle CBG$ (alt. ∠s, JI // BC)
$\angle IGJ = \angle BGC$ (vert. opp. ∠s)
$\angle IJG = \angle BCG$ (alt. ∠s, JI // BC)
$\therefore \Delta GIJ \sim \Delta GBC$ (Equiangular)
$\frac{GI}{BG} = \frac{GJ}{CG} = \frac{JI}{BC} = 2 \quad \cdots(6)$ (corr. sides, ~ Δs) and by (5)

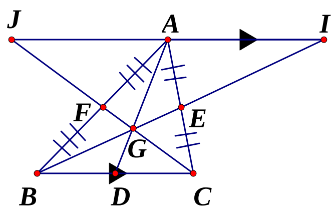
$\frac{2BE}{BG} = \frac{BI}{BG} = \frac{BG+GI}{BG} = 1 + \frac{GI}{BG} = 1+2 = 3$ by (2) and (6)

$\Rightarrow \frac{BE}{BG} = \frac{3}{2} \Rightarrow \frac{BG+GE}{BG} = \frac{3}{2} \Rightarrow 1 + \frac{GE}{BG} = \frac{3}{2} \Rightarrow \frac{GE}{BG} = \frac{1}{2} \Rightarrow BG : GE = 2 : 1 \cdots(7)$

$\frac{2CF}{CG} = \frac{CJ}{CG} = \frac{CG+GJ}{CG} = 1 + \frac{GJ}{CG} = 1+2 = 3$ by (4) and (6)

$\Rightarrow \frac{CF}{CG} = \frac{3}{2} \Rightarrow \frac{CG+GF}{CG} = \frac{3}{2} \Rightarrow 1 + \frac{GF}{CG} = \frac{3}{2} \Rightarrow \frac{GF}{CG} = \frac{1}{2} \Rightarrow CG : GF = 2 : 1 \cdots(8)$

Join AG and produce AG to cut BC at D.



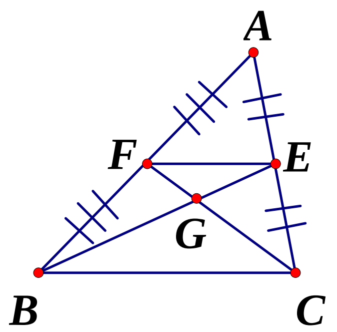
$\angle IAG = \angle BDG$ (alt. ∠s, JI // BC)
$\angle AGI = \angle DGB$ (vert. opp. ∠s)
$\angle AIG = \angle DBG$ (alt. ∠s, JI // BC)
$\therefore \Delta AGI \sim \Delta DGB$ (Equiangular)
$\frac{BD}{AI} = \frac{GD}{AG} = \frac{BG}{GI}$ (corr. sides ~ Δs)

$$\frac{BG}{GI} = \frac{\frac{2}{3}BE}{GE+EI} = \frac{\frac{2}{3}BE}{\frac{1}{3}BE+BE} = \frac{\frac{2}{3}}{\frac{4}{3}} = \frac{1}{2}$$

$\therefore AG : GD = 2 : 1$ and $BD = \frac{1}{2}GI = \frac{1}{2}BC$

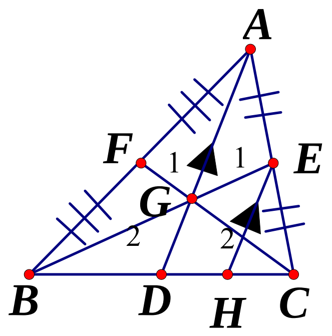
$\therefore AD$ is a median
The 3 **medians** are **concurrent** at the **centroid** G.

Method 3 To prove that the 3 medians are **concurrent** at the **centroid**.
Suppose the medians BE and CF intersects at G.
Try to show that the line AG produced cuts BC at D is a median.
Join FE.



$\frac{AF}{AB}=\frac{1}{2}=\frac{AE}{AC}$ (∵ E, F are the mid-points)
$\angle EAF=\angle CAB$ (common ∠s)
$\therefore \Delta AEF \sim \Delta ACB$ (2 sides proportional, included ∠)
$\angle AFE=\angle ABC$ (corr. ∠s, ~ Δs)
FE // BC …… (1) (corr. ∠s eq.)
$\frac{FE}{BC}=\frac{AF}{AB}=\frac{1}{2}$ …… (2) (corr. sides, ~ Δs)
$\angle GEF=\angle GBC$ (alt. ∠s, FE // BC)
$\angle EGF=\angle BGC$ (vert. opp. ∠s)
$\angle GFE=\angle GCB$ (alt. ∠s, FE // BC)
$\therefore \Delta GEF \sim \Delta GBC$ (equiangular)
$\frac{FG}{GC}=\frac{EG}{GB}=\frac{FE}{BC}=\frac{1}{2}$ …… (3) (corr. sides, ~ Δs) and by (2)

Join AG and produce AG to cut BC at D.
Through E draw a line parallel to AD cutting BC at H.



$\angle DAC=\angle HEC$ (corr. ∠s, AD // EH)
$\angle ACD=\angle ECH$ (common ∠s)
$\angle ADC=\angle EHC$ (corr. ∠s, AD // EH)
$\therefore \Delta ACD \sim \Delta ECH$ (Equiangular)
$\frac{AD}{EH}=\frac{CD}{CH}=\frac{AC}{EC}=2$ …… (4) (corr. sides ~Δs, E is the mid-point)
$\Rightarrow CD=2CH$ …… (5)

In ΔBDG and ΔBHE
$\angle BGD=\angle BEH$ (corr. ∠s, AD // EH)
$\angle GBD=\angle EBH$ (common ∠s)
$\angle BDG=\angle BHE$ (corr. ∠s, AD // EH)
$\therefore \Delta BGD \sim \Delta BEH$ (Equiangular)
$\frac{BD}{BH}=\frac{GD}{EH}=\frac{BG}{BE}=\frac{2}{3}$ …… (6) (corr. sides ~Δs, E is the mid-point) and by (3)

(6)÷(4) $\frac{GD}{EH}\div\frac{AD}{EH}=\frac{2}{3}\div 2$

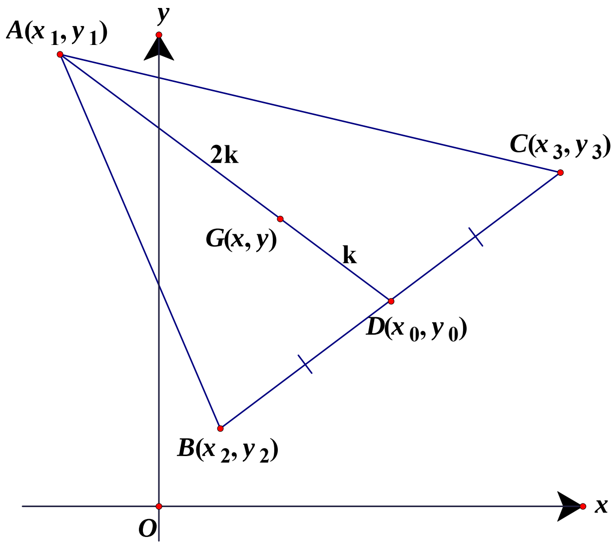
$\Rightarrow \frac{GD}{AD}=\frac{1}{3} \Rightarrow AG:GD=2:1$ …… (7)

From (6), $BD=\frac{2}{3}BH=\frac{2}{3}(BD+DH)$

$\Rightarrow \frac{1}{3}BD=\frac{2}{3}DH \Rightarrow BD=2DH$ …… (8)

By (4), $CD=2CH$ …… (9)
(8) + (9) $BC=BD+CD=2(DH+CH)=2CD$
$\Rightarrow D$ is the mid-point of BC
$\Rightarrow AD$ is the median
$\Rightarrow$ The 3 **medians** are **concurrent** at the **centroid** G.

Introduce a rectangular coordinates system as shown. Let the coordinates of $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ respectively. Express the coordinates of the centroid G in terms of x_1, x_2, x_3, y_1, y_2 and y_3.



$D(x_0, y_0)$ is the mid-point of BC. $D = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$

$G(x, y)$ is the centroid. $AG : GD = 2 : 1$.

$$G = \left(\frac{x_1+2x_0}{2+1}, \frac{y_1+2y_0}{2+1}\right)$$

$$= \left(\frac{x_1+2\cdot\frac{x_2+x_3}{2}}{3}, \frac{y_1+2\cdot\frac{y_2+y_3}{2}}{3}\right)$$

$$G = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

Example Let the coordinates of A, B and C be (–2, 8), (2, 2), (9, 5).

Then the coordinates of the centroid $= \left(\frac{-2+2+9}{3}, \frac{8+2+5}{3}\right)$ = (3, 5).