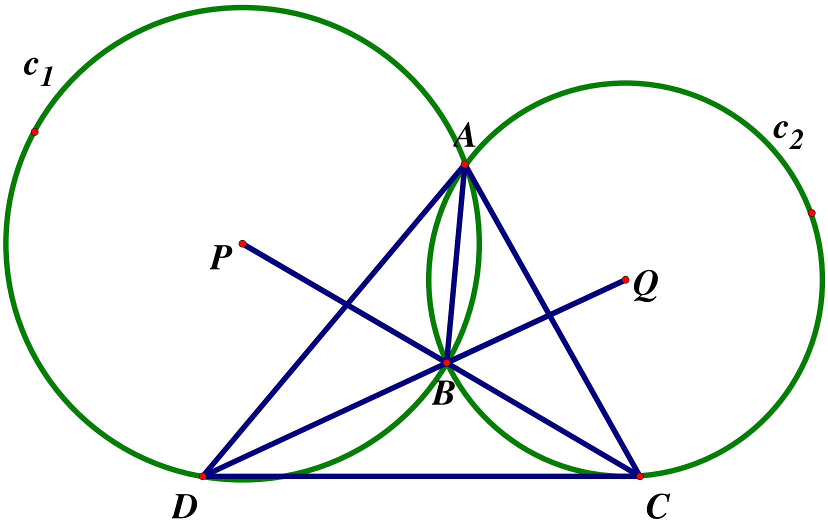
Incentre Example 1

Reference: 證明內心的技巧 **http://www.mathdb.org/resource_sharing/geometry/sc_incentres.doc**

Created by: Mr. Francis Hung Last updated: 3 September 2021



Let P and Q be the centres of circles C_1 and C_2 intersecting at two distinct points A and B.
If PB is extended to meet C_2 at C and QB is extended to meet C_1 at D, **prove that B is the incentre of ΔACD.**

Proof: Join PD and QC. Let $\angle BAD = \alpha$

$\angle BPD = 2\alpha$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$PD = PB$ (radii of circle C_1)

$\angle PBD = (180° - 2\alpha) \div 2 = 90° - \alpha$ ($\angle$ sum of Δ)

$\angle QBC = \angle PBD = 90° - \alpha$ (vert. opp. $\angle$s)

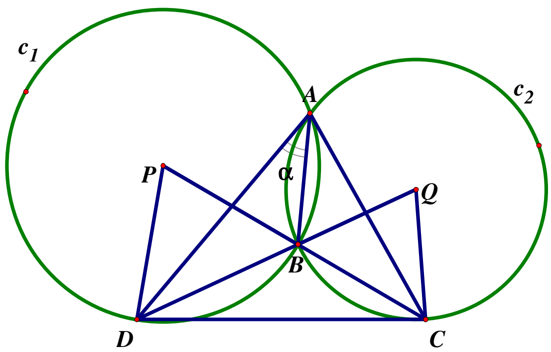
$QB = QC$ (radii of circle C_2)

$\angle BQC = 180° - 2(90° - \alpha) = 2\alpha$ ($\angle$ sum of Δ)

$\angle BAC = \alpha$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$\therefore \angle BAD = \angle BAC$

BA is the $\angle$ bisector of $\angle CAD$



Also, $\angle CPD = 2\alpha = \angle CQD = \angle CAD$ (proved)

$\therefore C, Q, A, P, D$ are concyclic

(converse, $\angle$s in the same segment)

$AP = PD$ (radii of circle C_1)

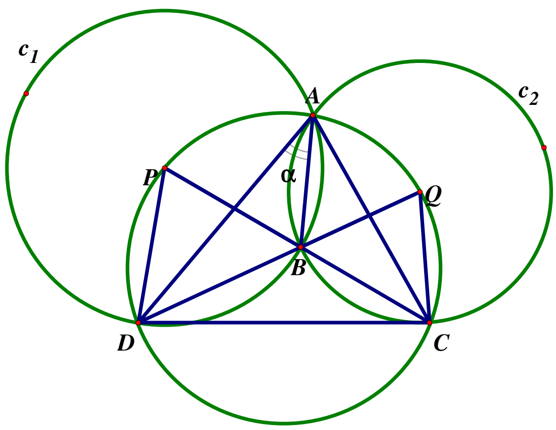
$\angle ACP = \angle DCP$ (eq. chords eq. $\angle$s)

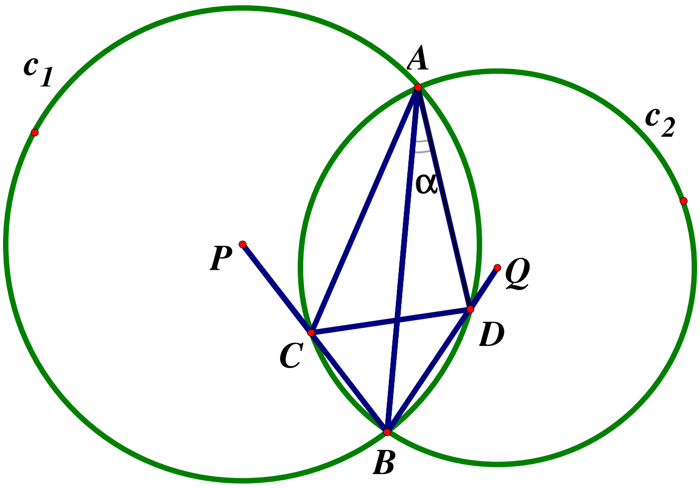
similarly $AQ = CQ$ (radii of C_2)

$\angle ADQ = \angle CDQ$ (eq. chords eq. $\angle$s)

$\therefore BC$ bisects $\angle ACD$ and BD bisects $\angle ADC$.

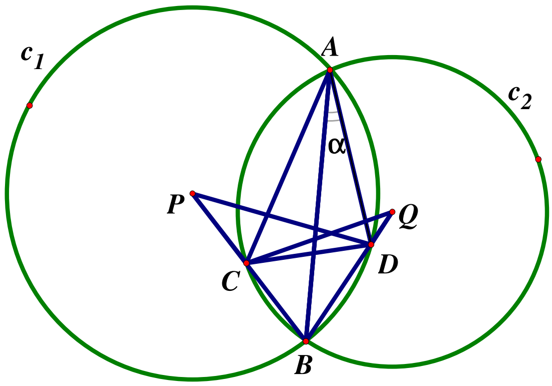
$\therefore B$ is the incentre of ΔACD.





Let P and Q be the centres of circles C_1 and C_2 intersecting at two distinct points A and B.

If PB intersects C_2 at C and QB intersects C_1 at D, prove that B is the excentre of ΔACD.



Proof: Join PD and QC. Let $\angle BAD = \alpha$

$\angle BPD = 2\alpha$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$PD = PB$ (radii of circle C_1)

$\angle PBD = (180° - 2\alpha) \div 2 = 90° - \alpha$ ($\angle$ sum of Δ)

$QB = QC$ (radii of circle C_2)

$\angle QBC = \angle QCB = 90° - \alpha$ (base $\angle$s isos. Δ)

$\angle BQC = 180° - 2(90° - \alpha) = 2\alpha$ ($\angle$ sum of Δ)

$\angle BAC = \alpha$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$\therefore \angle BAD = \angle BAC$

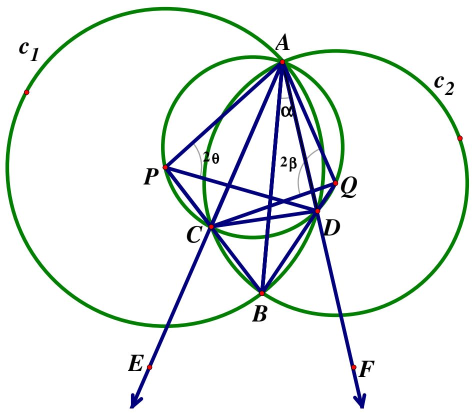
BA is the $\angle$ bisector of $\angle CAD$

Also, $\angle CPD = 2\alpha = \angle CQD = \angle CAD$ (proved)

$\therefore C, Q, A, P, D$ are concyclic

(converse, $\angle$s in the same seg.)

Produce AC to E and AD to F.



Let $\angle APD = 2\theta$, $\angle AQC = 2\beta$

$AP = PD$ (radii of circle C_1)

Let $\angle DAP = \angle ADP = 90° - \theta$ (base $\angle$s isos. Δ)

similarly $AQ = CQ$ (radii of C_2)

$\angle QAC = \angle QCA = 90° - \beta$ (base $\angle$s isos. Δ)

$\angle ECB = \angle ACP$ (vert. opp. $\angle$s)

$= \angle ADP = 90° - \theta$ ($\angle$s in the same seg.)

$\angle BCD = \angle DAP = 90° - \theta$ (ext. $\angle$, cyclic quad.)

$\therefore \angle ECB = \angle BCD$

BC is the exterior angle bisector of $\angle ACD$.

$\angle FDB = \angle ADQ$ (vert. opp. $\angle$s)

$= \angle ACQ$ ($\angle$s in the same segment)

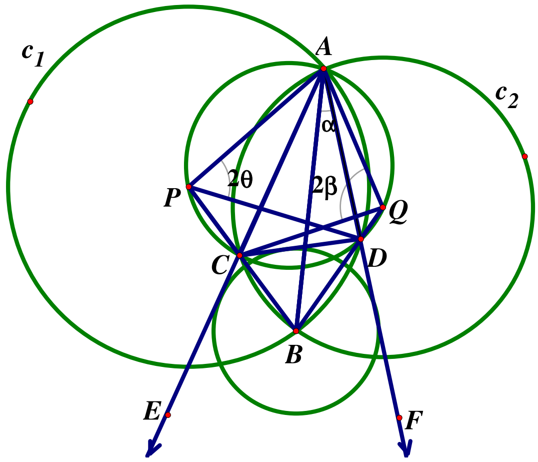
$= 90° - \beta$ (proved)

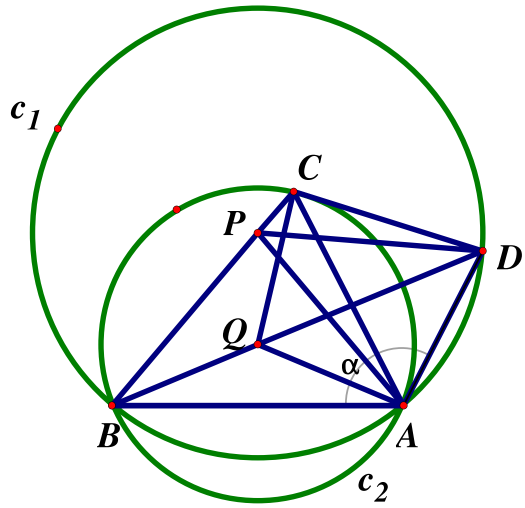
$\angle BDC = \angle CAQ = 90° - \beta$ (ext. $\angle$, cyclic quad.)

$\therefore \angle FDB = \angle BDC$

BD is the exterior angle bisector of $\angle ADC$.

$\therefore B$ is the excentre of ΔACD.





Let P and Q be the centres of circles C_1 and C_2 intersecting at two distinct point A and B.
If BP is extended to meet C_2 at C and BQ is extended to meet C_1 at D, prove that B is the excentre of ΔACD.

Proof: Join PD and QC. Let $\angle BAD = \alpha$

reflex $\angle BPD = 2\alpha$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$\angle BPD = 360° - 2\alpha$ ($\angle$s at a point)

$PD = PB$ (radii of circle C_1)

$\angle PBD = [180° - (360° - 2\alpha)] \div 2 = \alpha - 90°$ ($\angle$ sum of Δ)

$QB = QC$ (radii of circle C_2)

$\angle QBC = \angle QCB = \alpha - 90°$ (base $\angle$s isos. Δ)

$\angle BQC = 180° - 2(\alpha - 90°) = 360° - 2\alpha$ ($\angle$ sum of Δ)

$\angle BAC = 180° - \alpha$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

Produce DA to E.

$\angle BAE = 180° - \alpha$ (adj. $\angle$s on st. line)

$\therefore \angle BAE = \angle BAC$

BA is the exterior $\angle$ bisector of $\angle CAD$.

$\angle CAD = \alpha - \angle BAC = 2\alpha - 180°$

$\angle CPD = 180° - \angle BPD = 2\alpha - 180°$ (adj. $\angle$s on st. line)

$\angle CQD = 180° - \angle BQC = 2\alpha - 180°$ (adj. $\angle$s on st. line)

$\therefore \angle CPD = \angle CQD = \angle CAD$

$\therefore C, Q, A, P, D$ are concyclic

(converse, $\angle$s in the same segment)

$CQ = AQ$ (radii of circle C_2)

$\angle CQD = \angle ADQ$ (eq. chords eq. $\angle$s)

$\therefore BD$ bisects $\angle ADC$.

B is the excentre of ΔACD.