Circumcircle

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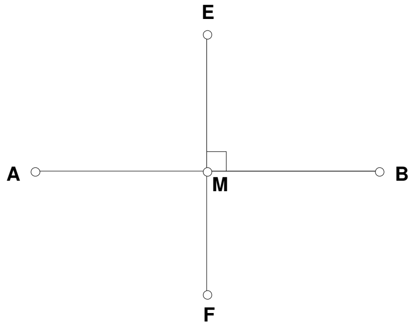
The **perpendicular bisector** of a line segment.



Given a line segment AB.

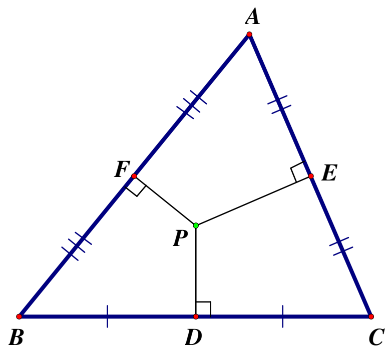
A line segment EF intersects AB at M.

If (1) $EF \perp AB$ and (2) $AM = MB$;

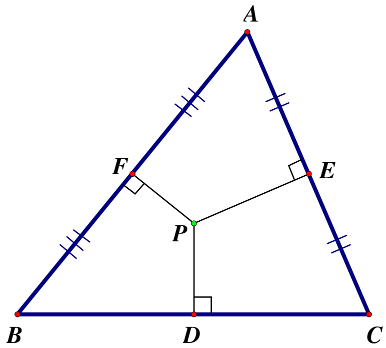
then EF is called the perpendicular bisector of AB.

Theorem The three perpendicular bisectors of a triangle are **concurrent** at a point P called the **circumcentre**. PD, PE and PF are perpendicular bisectors of BC, CA and AB respectively.

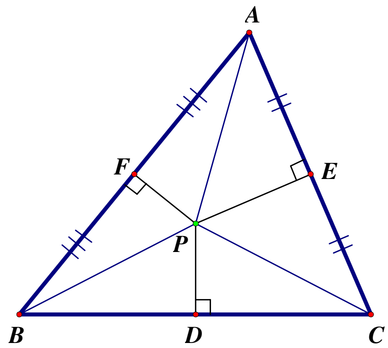
The theorem says that PD, PE and PF meet at a point P.



Proof: Let PE and PF be the two perpendicular bisectors of AC and AB respectively which intersect at P.
Through P draw a line segment PD perpendicular to BC.
Try to show that PD is a perpendicular bisector of BC.
Join AP, BP and CP.



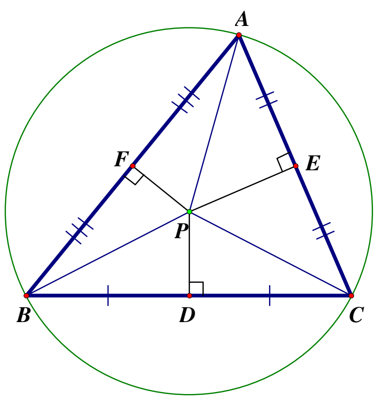
$PE = PE$	(common sides)
$AE = EC$	(PE is a $\perp$ bisector)
$\angle AEP = \angle CEP = 90°$	(PE is a $\perp$ bisector)
$\Delta APE \cong \Delta CPE$	(S.A.S.)
$PF = PF$	(common sides)
$AF = FB$	(PF is a $\perp$ bisector)
$\angle AFP = \angle BFP = 90°$	(PF is a $\perp$ bisector)
$\Delta APF \cong \Delta BPF$	(S.A.S.)
$\therefore BP = AP = CP$	(corr. sides, $\cong \Delta$s)
$PD = PD$	(common side)
$\angle BDP = 90° = \angle CDP$	(by construction)
$\Delta BDP \cong \Delta CDP$	(R.H.S.)
$\therefore BD = CD$	(corr. sides, $\cong \Delta$s)



$\therefore PD$ is a perpendicular bisector of BC.
The theorem is proved.

$\because AP = BP = CP$

We can use P as centre and $AP = BP = CP$ as radius to draw a circle which passes through the triangle ABC. The circle is called the **circumscribed circle** (or circum-circle in short form). The centre is called the circumscribed centre (or circum-centre in short) and the radius is called the circum-radius.



Let $\angle PAE = \angle PCE = x$, $\angle PAF = \angle PBF = y$, $\angle PBD = \angle PCD = z$ (corr. ∠s, ≅ Δs)

$2x + 2y + 2z = 180°$ (∠ sum of ΔABC)

$180° - 2z = 2(x + y)$

$\angle BPC = 180° - 2z$ (∠ sum of ΔBPC)

$= 2(x + y)$

$= 2\angle A$

$\because \Delta BDP \cong \Delta CDP$

$\therefore \angle BPD = \angle CPD = \angle A$

Let the circumscribed radius be R.

In ΔBPD, $\dfrac{BD}{BP} = \sin \angle BPD$

$\Rightarrow \dfrac{\frac{a}{2}}{R} = \sin A$

$\Rightarrow \dfrac{a}{\sin A} = 2R$

In a similar manner we can prove that $\dfrac{b}{\sin B} = 2R$; $\dfrac{c}{\sin C} = 2R$.

Therefore, we have proved the **Sine formula** $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$;

where R is the radius of the circumscribed circle.

By Heron's formula the area of $\Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$

where $s = \dfrac{1}{2}(a + b + c)$ (Half of the perimeter of ΔABC).

$\Rightarrow \sqrt{s(s-a)(s-b)(s-c)} = \dfrac{1}{2}ab\sin C$

$\Rightarrow \sqrt{s(s-a)(s-b)(s-c)} = \dfrac{1}{2}ab\dfrac{c}{2R}$

$\Rightarrow R = \dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$

This is the formula of the radius in terms of the sides of triangle.

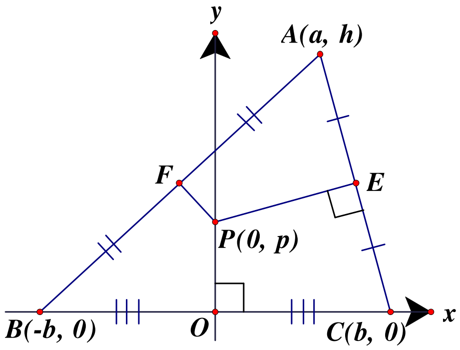
Method 2 (Coordinates)

Use analytic approach to prove that the 3 ⊥ bisectors of ΔABC are concurrent at the circentre P.
Define a rectangular coordinates system with BC lying on the x-axis.
Let the coordinates of B and C be $(-b, 0)$ and $C(b, 0)$ respectively.

$\because OB = OC = b$

$\therefore O$ is the mid-point of BC

x-axis $\perp$ y-axis $\Rightarrow$ y-axis is the $\perp$ bisector of BC.
Let the coordinates of A be (a, h).



Let E and F be the mid-points of AC and AB respectively.
Suppose the perpendicular bisector of AC cuts y-axis at $P(0, p)$.

$E = \left(\frac{a+b}{2}, \frac{h}{2}\right)$, $F = \left(\frac{a-b}{2}, \frac{h}{2}\right)$

$\because PE \perp AC$

$$\therefore m_{PE} \times m_{AC} = \frac{\frac{h}{2} - p}{\frac{a+b}{2}} \times \frac{h}{a-b} = -1 \Rightarrow \frac{2\left(\frac{h}{2} - p\right)h}{(a+b)(a-b)} = -1$$

$$m_{PF} \times m_{AB} = \frac{\frac{h}{2} - p}{\frac{a-b}{2}} \times \frac{h}{a+b} = -1 \Rightarrow \frac{2\left(\frac{h}{2} - p\right)h}{(a-b)(a+b)} = -1$$

$\therefore$ The $PF \perp AB$

The 3 ⊥ bisectors of a triangle are concurrent at the circumcentre P.