Orthocentre

Created by Francis Hung Last updated: 26 September 2021

The **three altitudes** of a **triangle** are **concurrent** at a point called the "**orthocentre**". (Figure 1)

In ΔABC, $AD \perp BC$, $BE \perp AC$, $CF \perp AB$.

Then AD, BE, CF are **concurrent** at H.

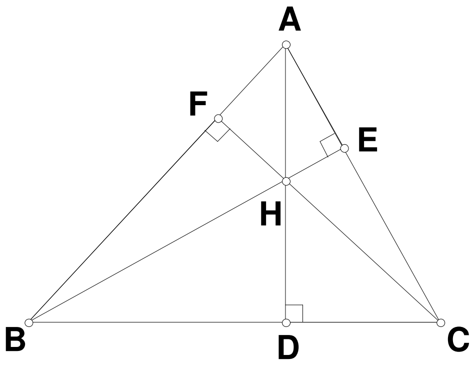


Figure 1

Proof: Let the **altitudes** BE and CF meet at H.

Join AH and produce it to meet BC at D.

Try **to show** that $AD \perp BC$. (Figure 2)

$\angle AFH + \angle AEH = 180°$

A, F, H, E are concyclic. (opp. $\angle$ supp.)

$\angle BFC = \angle BEC$

B, C, E, F are concyclic. (converse, $\angle$s in the same seg.)

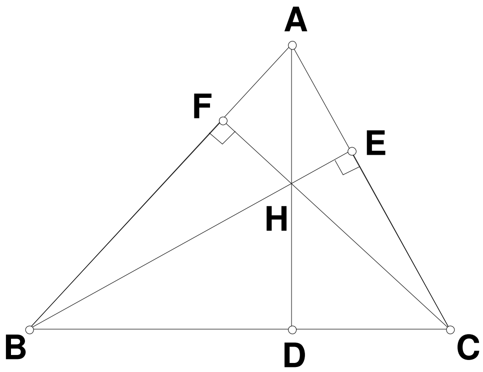


Figure 2

In **Figure 3**, let $\angle BAD = x$, $\angle AHF = y$.

$\angle FEH = x$ ($\angle$s in the same seg.)

$\angle BCF = \angle BEF$ ($\angle$s in the same seg.)

$= x$

$\angle CHD = y$ (vert. opp. $\angle$s.)

In ΔAFH, $x + y = 90°$ ($\angle$ sum of Δ)

In ΔCDH, $x + y + \angle CDH = 180°$ ($\angle$ sum of Δ)

$\therefore \angle CDH = 90°$

The **theorem is proved.**

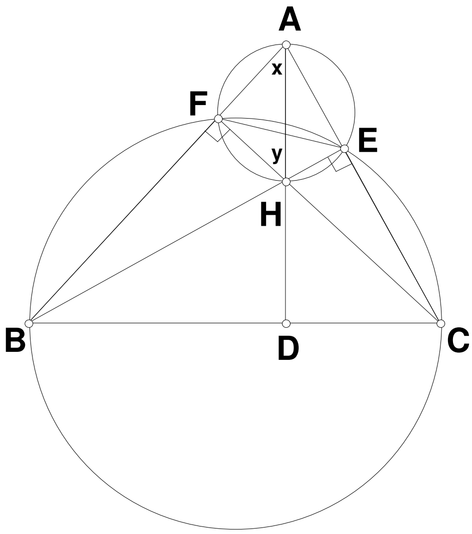
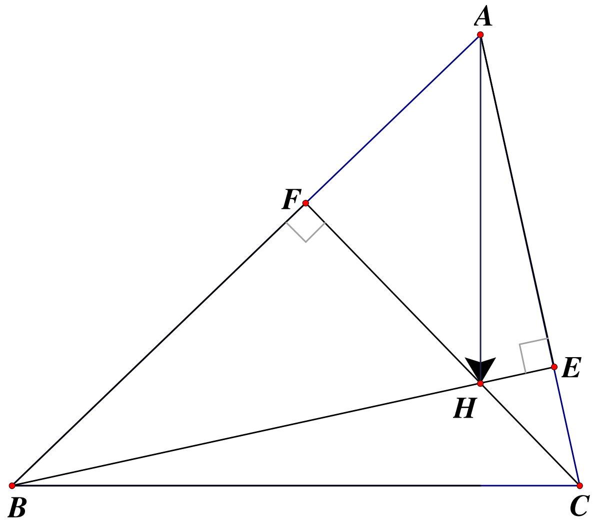


Figure 3

Vector method



Let O be the reference point.
Suppose the altitudes BE and CF intersect at H.
Let the position vectors of A, B, C, H be $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{h}$.
Join AH. We are going to show that $\overrightarrow{AH}\cdot\overrightarrow{BC}=0$

$\overrightarrow{BH}=\mathbf{h}-\mathbf{b}$, $\overrightarrow{CH}=\mathbf{h}-\mathbf{c}$

$\because$ $BH\perp AC$ and $CH\perp AB$

$\therefore$ $(\mathbf{h}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{a})=0$ and $(\mathbf{h}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{a})=0$

$\mathbf{h}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{c}-\mathbf{h}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}=0$ …… (1)

$\mathbf{h}\cdot\mathbf{b}-\mathbf{c}\cdot\mathbf{b}-\mathbf{h}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{c}=0$ …… (2)

(1) – (2): $\mathbf{h}\cdot\mathbf{c}-\mathbf{h}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{c}=0$

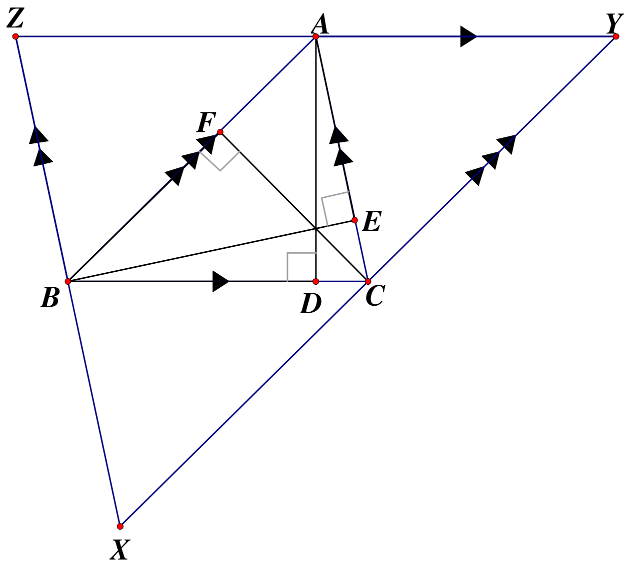
$(\mathbf{h}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{b})=0$

$\therefore$ $\overrightarrow{AH}\cdot\overrightarrow{BC}=0$

$AH\perp BC$

i.e. The 3 altitudes are concurrent at a point H.

Method 3 (The following method use the fact that the 3 perpendicular bisectors of a triangle are concurrent at the circumcentre)
(Reference: New Trend Mathematics S.3B, Chung Tai Educational Press, 2003, p.167-168)



Construct three straight lines ZAY, ZBX and YCX such that ZAY // BC, ZBX // AC and YCX // AB.

In ΔABC and ΔXCB,

$\because$ AC // BX and AB // CX	(by construction)
$\therefore$ $\angle ACB=\angle XBC$	(alt. $\angle$s, AC // BX)
$\angle ABC=\angle XCB$	(alt. $\angle$s, AB // CX)
$BC=CB$	(common side)
$\therefore$ $\Delta ABC\cong\Delta XCB$	(A.S.A.)
$\therefore$ $CA=BX$	(corr. sides, $\cong\Delta$s)

In ΔZAB and ΔCBA,

$\because$ ZA // BC and ZB // AC	(by construction)
$\therefore$ $\angle ZAB=\angle CBA$	(alt. $\angle$s, ZA // BC)
$\angle ZBA=\angle CAB$	(alt. $\angle$s, ZB // AC)
$AB=BA$	(common side)
$\therefore$ $\Delta ZAB\cong\Delta CBA$	(A.S.A.)
$\therefore$ $ZB=CA$	(corr. sides, $\cong\Delta$s)
$\therefore$ $ZB=BX$	
$\because$ AC // ZX	(by construction)
$\therefore$ $\angle AEB+\angle EBZ=180^\circ$	(int. $\angle$s, AC // ZX)
$90^\circ+\angle EBZ=180^\circ$	
$\angle EBZ=90^\circ$	

$\therefore$ BE is the perpendicular bisector of ZX.
Similarly, AD is the perpendicular bisector of ZY.
CF is the perpendicular bisector of YX.

In ΔXYZ,

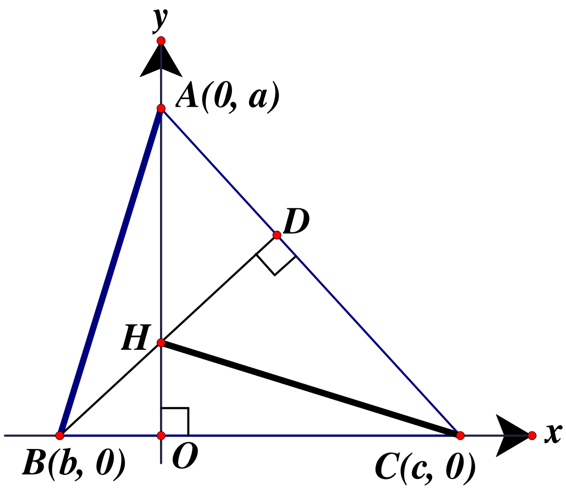
$\because$ three perpendicular bisectors intersect at a point. (proved)

$\therefore$ AD, BE and CF intersect at a point.

Method 4 (Coordinates)

Prove that the 3 altitudes are concurrent at the orthocentre H.

Define a rectangular coordinates system with BC lying on x-axis, $OA \perp x$-axis.
Let the coordinates of A, B and C be $(0, a)$, $(b, 0)$ and $(c, 0)$.



x-axis $\perp$ y-axis $\Rightarrow OA \perp BC \Rightarrow OA$ is an altitude.
Suppose another altitude BD intersects OA at H. i.e. $BD \perp AC$
Let the coordinates of H be $(0, h)$.

$m_{BD} \times m_{AC} = -1$

$\frac{h-0}{0-b} \times \frac{a-0}{0-c} = -1 \Rightarrow \frac{ah}{bc} = -1$

$m_{CH} \times m_{AB} = \frac{h-0}{0-c} \times \frac{a-0}{0-b} = \frac{ah}{bc} = -1$

$\therefore CH \perp AB$

$\therefore$ The 3 altitudes are concurrent at the orthocentre $H(0, h)$.