Problem on a 10°, 10°, 40° triangle

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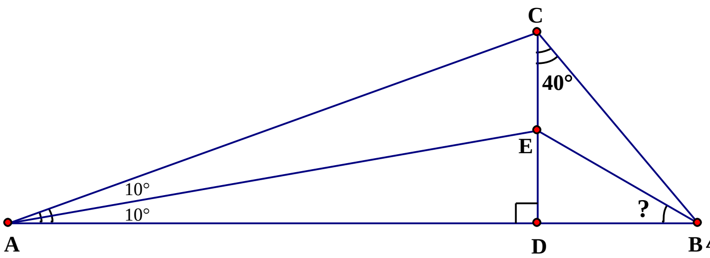

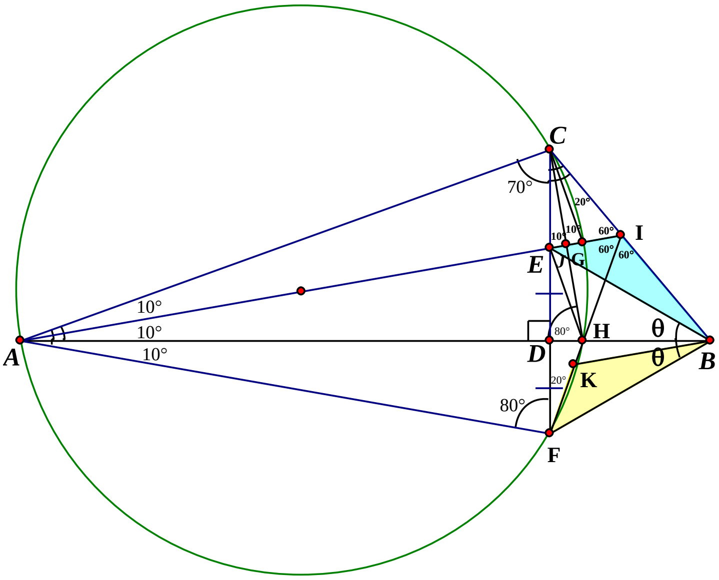

In the figure, ADB is a straight line. The line segment $CED \perp AB$. $\angle DAE = \angle CAE = 10°$. $\angle BCD = 40°$. Find $\angle DBE$.

Solution: Let $\angle ABE = \theta$. Reflect ΔAEB along AB to ΔAFB.

$\Delta AEB \cong \Delta AFB$	(by definition)
$BE = BF$	(corr. sides, $\cong\Delta$s)
$\angle ABE = \angle ABF = \theta$	(corr. $\angle$s, $\cong\Delta$s)
$\angle BAF = \angle BAE = 10°$	(corr. $\angle$s, $\cong\Delta$s)
$AF = AE$	(corr. sides, $\cong\Delta$s)
$AD = AD$	(common sides)
$\Delta ADE \cong \Delta ADF$	(S.A.S.)
$DE = DF$	(corr. sides, $\cong\Delta$s)
$\angle ACD = 70°$, $\angle AFD = 80°$	($\angle$ sum of ΔACD and ΔADF)
Construct the circumscribed circle passing through ACF	(by using perpendicular bisectors)
AB cuts the circle again at H. Join CH and FH.	
$\angle CAF + \angle CHF = 180°$	(opp. $\angle$s, cyclic quad.)
$\angle CHF = 180° - 30° = 150°$ ⋯⋯ (1)	
Extend AE to cut the circle at G and BC at I. Suppose CH intersects AI at J.	
$\angle FCH = \angle FAH = 10°$	($\angle$s in the same segment)
$\angle GCH = \angle GAH = 10°$	($\angle$s in the same segment)
$\angle GCI = \angle BCD - \angle DCH - \angle GCH = 40° - 10° - 10° = 20°$	
$\angle CAH + \angle ACH + \angle AHC = 180°$	($\angle$ sum of ΔACH)
$\angle AHC = 180° - (10° + 10°) - (70° + 10°) = 80°$	
$\therefore\ \angle ACJ = \angle AHJ = 80°$	
$\angle CAJ = \angle BAJ = 10°$	(given)
$AJ = AJ$	(common side)
$\therefore\ \Delta ACJ \cong \Delta ABJ$	(A.A.S.)
$\angle AJC = \angle AJH$	(corr. $\angle$s, $\cong\Delta$s)
$\angle AJC + \angle AJH = 180°$	(adj. $\angle$s on st. line)
$\therefore\ \angle AJC = \angle AJH = 90°$	
$CJ = HJ$	(corr. sides, $\cong\Delta$s)
$IJ = IJ$	(common side)
$\therefore\ \Delta CIJ \cong \Delta HIJ$	(S.A.S.)
$\angle IHJ = \angle ICJ = 30°$ ⋯⋯ (2)	(corr. $\angle$s, $\cong\Delta$s)

$\angle CHF + \angle IHJ = 150° + 30° = 180°$ (by (1) and (2))
$\therefore$ I, H, F are collinear
$\angle CBD = 180° - 90° - 40° = 50°$ ($\angle$ sum of ΔBCD)
$\angle AIC = 180° - 10° - (70° + 40°) = 60°$ ($\angle$ sum of ΔACI)
$\angle CFI = \angle CFH = \angle CAH = 20°$ ($\angle$s in the same segment)
$\angle CIF = 180° - 20° - 40° = 120°$ ($\angle$ sum of ΔCFI)
$\angle EIF = \angle CIF - \angle AIC = 120° - 60° = 60°$
$\angle BIE = 180° - 60° = 120°$ (adj. $\angle$s on st. line)
$\angle BIF = 180° - 120° = 60°$ (adj. $\angle$s on st. line)
Locate a point K on IF such that $IK = IB$. Then, by definition, ΔIKB is an isosceles Δ.
$\angle IKB = \angle IBK$ (base $\angle$s, isos. Δ)
$= \dfrac{180° - 60°}{2}$ ($\angle$ sum of ΔBIK)
$= 60°$
$\therefore$ ΔBIK is an equilateral triangle
$BI = BK \cdots\cdots (3)$ (property of equilateral triangle)
$BE = BF \cdots\cdots (4)$ (proved, ΔAFB is the reflected image of ΔAEB)
$\angle BKF = 180° - 60° = 120°$ (adj. $\angle$s on st. line)
$\dfrac{BF}{\sin\angle BKF} = \dfrac{BK}{\sin\angle BFK} \Rightarrow \dfrac{BF}{\sin 120°} = \dfrac{BK}{\sin\angle BFK} \cdots\cdots (5)$ (sine rule on ΔBFK)
$\dfrac{BE}{\sin\angle BIE} = \dfrac{BI}{\sin\angle BEI} \Rightarrow \dfrac{BE}{\sin 120°} = \dfrac{BI}{\sin\angle BEI} \cdots\cdots (6)$ (sine rule on ΔBEI)
(5)÷(6): $1 = \dfrac{\sin\angle BFK}{\sin\angle BEI}$ (by (3) and (4))
$\therefore$ $\angle BFK = \angle BEI$ ($\because$ $\angle BIE = \angle BKF = 120° > 90°$)
$\therefore$ B, I, E, F are concylic (converse, $\angle$s in the same segment)
$\angle EBF = \angle EIF$ ($\angle$s in the same segment)
$2\theta = 60°$
$\theta = 30°$

Method 2 (provided by Maths De Mon in Facebook)

Let $AD = x$, $\angle DBE = \theta$
In ΔACD, $CD = x \tan 20°$; in ΔADE, $DE = x \tan 10°$
In ΔBCD, $BD = CD \tan 40° = x \tan 20° \tan 40°$

In ΔBDE, $\tan\theta = \dfrac{DE}{BD} = \dfrac{x\tan 10°}{x\tan 20° \tan 40°} = \dfrac{\tan 10°}{\tan(30° - 10°)\tan(30° + 10°)}$

$$\tan\theta = \frac{\tan 10°}{\dfrac{\tan 30° - \tan 10°}{1 + \tan 30° \tan 10°} \cdot \dfrac{\tan 30° + \tan 10°}{1 - \tan 30° \tan 10°}} = \frac{\tan 10°}{\dfrac{\tan^2 30° - \tan^2 10°}{1 - \tan^2 30° \tan^2 10°}} = \frac{\tan 10°}{\dfrac{\left(\frac{1}{\sqrt{3}}\right)^2 - \tan^2 10°}{1 - \left(\frac{1}{\sqrt{3}}\right)^2 \tan^2 10°}}$$

$$= \frac{\tan 10°\left(1 - \frac{1}{3}\tan^2 10°\right)}{\frac{1}{3} - \tan^2 10°} = \frac{\tan 10°\left(3 - \tan^2 10°\right)}{1 - 3\tan^2 10°} = \frac{3\tan 10° - \tan^3 10°}{1 - 3\tan^2 10°} = \tan 30°$$

$\theta = 30°$