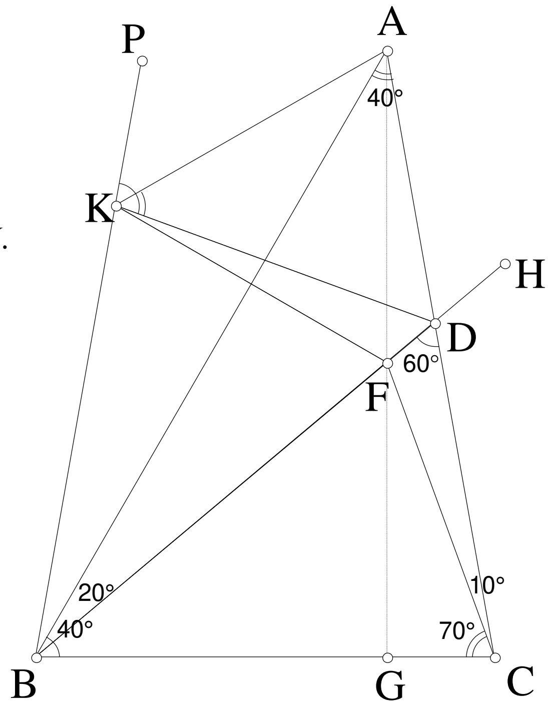
40°-60°-80° triangle

IMO (HK) Preliminary Selection Contest 2001-02 Q7

Created by Francis Hung   Last updated: 02 September 2021

In ΔABC, $\angle BAC = 40°$ and $\angle ABC = 60°$. D and E are points on sides AC and AB respectively such that $\angle CBD = 40°$ and $\angle BCE = 70°$. Let BD intersects CE at F and AF intersect BC at G. Find $\angle GFC$.

Solution:



Let CE cuts BD at F.

We don't draw FE as it is useless.

Produce BD to H.

$\angle ADH = 60°$ vert. opp. $\angle$s

Reflect the whole ΔBDC to the other side of BH.

(with BH as the line of symmetry.)

Let K be the image of C.

Join FK.

Since $\angle KBF = \angle FBC = 40°$

$\therefore \angle KBA = 20°$

$\angle KDB = \angle BDC = 60°$

$\therefore \angle ADK = 60° = \angle ADH$

$\therefore AB$ is the interior angle bisector of ΔBDK.

AD is the exterior angle bisector of ΔBDK.

$\therefore A$ is the ex-centre of ΔBDK.

Since $\angle BKD = \angle BCD = 80°$

$\therefore KA$ is also an exterior angle bisector of ΔBDK.

$\angle AKP = \angle AKD = \frac{1}{2}\left(180^\circ - 80^\circ\right) = 50°$

$\because K$ is the image of C

$\therefore \angle DKF = \angle DCF = 10°$ (corr. $\angle$s. $\cong\Delta$s, A.A.S.)

$\therefore \angle AKF = 50° + 10° = 60° = \angle ADH$

$\therefore ADFK$ is a cyclic quadrilateral (ext $\angle$ = int. opp. $\angle$)

$\therefore \angle FAD = \angle FKD = 10°$ ($\angle$s in the same segment)

$\angle GFC = \angle FAC + \angle FCA = 10° + 10° = 20°$ (ext. $\angle$ of Δ)