Equal angle bisectors

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Given ΔABC. If the two angle bisectors are equal in length, then it is isosceles.

Method 1

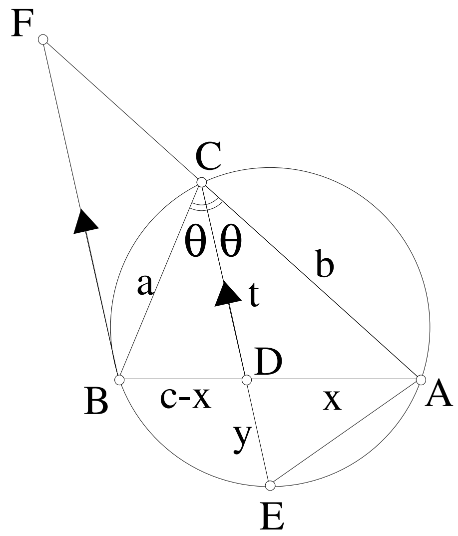
Let $BC = a$, $AC = b$, $AB = c$,

$CD = t$ = angle bisector of $\angle BCA$.

Draw a circle passes through ΔABC.

Produce CD at E on the circle.

Let $DE = y$



$\because \Delta BDC \sim \Delta ADE$ (equiangular)

$\therefore ty = (c - x)x \quad \cdots\cdots \cdots \quad (1)$

$\angle BCD = \angle ECA = \theta$ (given CD = angle bisector)

$\angle CBD = \angle CEA$ ($\angle$ in the same segment)

$\therefore \Delta CDB \sim \Delta CEA$ (equiangular)

$\therefore \dfrac{a}{t+y} = \dfrac{t}{b}$

$ab = t^2 + ty$

$t^2 = ab - ty$

$\quad = ab - (c - x)x \quad$ by (1) $\quad \cdots\cdots\cdots (2)$

Draw $BF // CD$, produce AC to meet at F.

$\angle CBF = \angle BCD = \theta$ (alt. $\angle$s, $BF // CD$)

$\angle BFC = \angle DCA = \theta$ (corr. $\angle$s, $BF // CD$)

$\therefore \Delta CBF$ is isosceles.

$\therefore CF = CB = a$

$\Delta CAD \sim \Delta FAB$ (equiangular)

$\therefore \dfrac{b}{a+b} = \dfrac{x}{c}$

$x = \dfrac{bc}{a+b} \quad \cdots\cdots\cdots (3)$

Sub. (3) into (2)

$$t^2 = ab - \left(c - \frac{bc}{a+b}\right)\frac{bc}{a+b} = \frac{(a+b+c)(a+b-c)ab}{(a+b)^2}$$

Let u = another angle bisector such that $t = u$

Similarly $u^2 = \dfrac{(a+b+c)(a-b+c)ac}{(a+c)^2}$

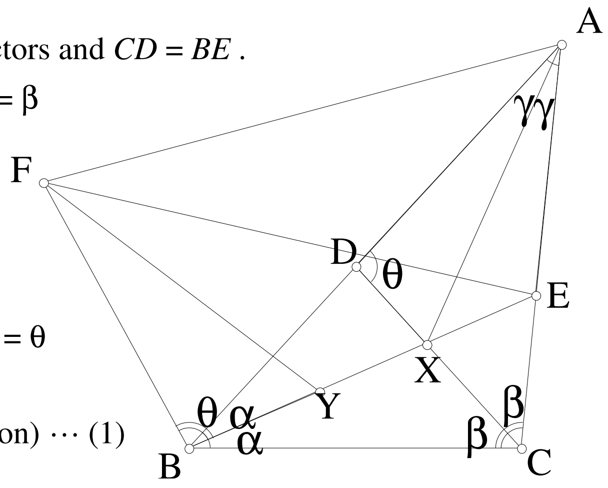
$t = u \Rightarrow [a^3 + a^2(b + c) + 3abc + bc(b + c)](b - c) = 0$

$b - c = 0$ ($\because$ the other factor is positive)

$b = c$

The triangle is isosceles.

Method 2



In the figure, let CD, BE be the angle bisectors and $CD = BE$.

Let $\angle CBE = \angle ABE = \alpha$, $\angle BCD = \angle ACD = \beta$

Suppose CD intersects BE at X. Join AX .

$\because$ the three angle bisectors are concurrent

$\therefore AX$ is the angle bisector.

Let $\angle BAX = \angle CAX = \gamma$, $\angle ADC = \theta$

Copy θ on BE so that $BF = AD$ and $\angle EBF = \theta$

Join EF

Then $\Delta EBF \cong \Delta CAD$ (S.A.S. by construction) $\cdots$ (1)

$\therefore \angle BFE = 2\gamma$

Draw FY bisects $\angle BFE$

$$\begin{cases} \angle YFE = \angle XAC = \gamma \\ \angle FEY = \angle ACX = \beta \quad \text{(by construction)} \\ \quad AC = FE \end{cases}$$

$\therefore$ $\Delta FEY \cong \Delta ACX$ (A.S.A.)

$\therefore$ $FY = AX$ (corr. sides of $\cong\Delta$s) $\cdots\cdots$ (2)

$\angle BFE = \angle BAE$ (by (1) $\Delta EBF \cong \Delta CAD$)

$\therefore$ F, A, E, B are concyclic (converse, $\angle$s in the same segment)

$\angle FAB = \angle FEB$ ($\angle$s in the same segment)

$= \angle ACD$ (by (1), $\Delta EBF \cong \Delta CAD$)

$= \beta$ $\cdots\cdots\cdots$ (*)

$\therefore$ $\angle FAX = \beta + \gamma$ $\cdots\cdots\cdots$ (3)

$\angle FYE = \angle FBY + \angle BFY$ (ext. $\angle$ of ΔBFY)

$= \theta + \gamma$ $\cdots\cdots\cdots$(4)

By considering the ΔAXD and ΔAXC

$\beta + \gamma + \theta + \gamma = 180° \Rightarrow \beta + \theta + 2\gamma = 180°$ $\cdots\cdots\cdots\cdots$(5)

(3) + (4) $\Rightarrow \angle FAX + \angle FYE = 180°$ (by (5))

$\therefore A$, X, Y, F are concyclic (opp. $\angle$s supp.) $\cdots\cdots\cdots\cdots$(6)

Join AY. Given $AX = FY$ (by (2))

$\angle FAY = \angle AYX$ (equal arcs, equal angles)

$\therefore AF$ // YX (alt. $\angle$s equal)

$\angle FAB = \angle ABE$ (alt. $\angle$ AF // YX)

$= \alpha$

By (*), $\beta = \alpha$

$\therefore \angle ABC = \angle ACB$

ΔABC is isosceles.