Equal and perpendicular segments in a square

Reference: https://www.cut-the-knot.org/m/Geometry/KvantM1202.shtml

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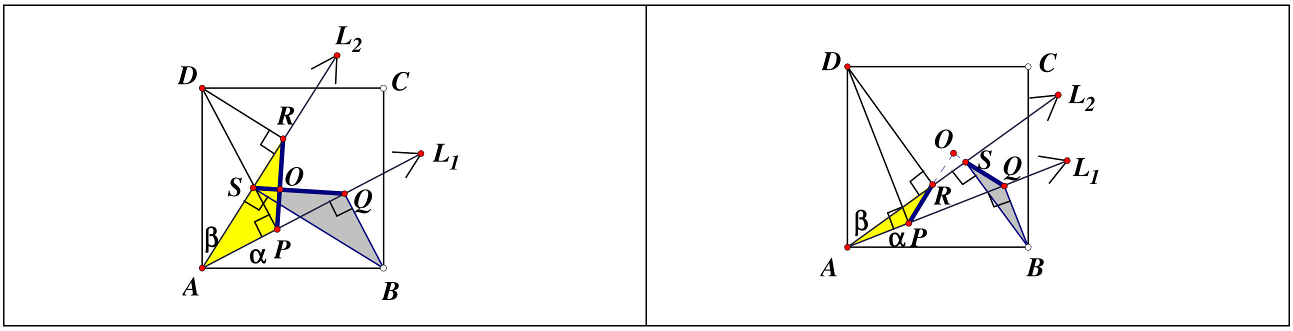

Given a square $ABCD$. Two rays L_1 and L_2 are drawn through A inside the square.

P, Q, R and S are the feet of perpendiculars from D and B onto L_1 and L_2 respectively.

Prove that $PR = QS$ and $PR \perp QS$.

Proof: Let $\angle BAQ = \alpha$, $\angle DAR = \beta$

$\angle DAQ = 90° - \alpha$, $\angle BAS = 90° - \beta$

$\angle ADP = 180° - \angle DAP - \angle DAP = 180° - 90° - (90° - \alpha) = \alpha$ ($\angle$ sum of Δ)

$\angle ABS = 180° - \angle BSA - \angle BAS = 180° - 90° - (90° - \beta) = \beta$ ($\angle$ sum of Δ)

$AB = AD$ (sides of a square)

$\angle AQB = \angle APD = \angle ASB = \angle ARD = 90°$

$\Delta AQB \cong \Delta DPA$, $\Delta ASB \cong \Delta DRA$ (A.A.S.)

$BQ = AP$, $BS = AR$ (corr. sides, $\cong \Delta$s)

$\angle PAR = 90° - \alpha - \beta$

$\angle ABS = 180° - 90° - (90° - \beta) = \beta$ ($\angle$ sum of Δ)

$\angle ABQ = 90° - \alpha$ ($\angle$ sum of Δ)

$\angle QBS = 90° - \alpha - \beta$

$\therefore\ \angle PAR = \angle QBS$

$\Delta QBS \cong \Delta PAR$ (A.A.S.)

$PR = QS$ (corr. sides, $\cong \Delta$s)

$\angle APR = \angle BQS$ (corr. $\angle$s, $\cong \Delta$s)

$\angle APS + \angle SPR = \angle BQP + \angle PQS$

$90° + \angle SPR = 90° + \angle PQS$

$\angle SPR = \angle PQS$

Suppose PR and QS intersect at O.

In ΔPOQ, $\angle PQO + \angle POQ = \angle APO$ (ext. $\angle$ of Δ)

$\angle POQ = 90° + \angle SPR - \angle PQO = 90°$

$\therefore\ PR \perp QS$

The proof is complete.