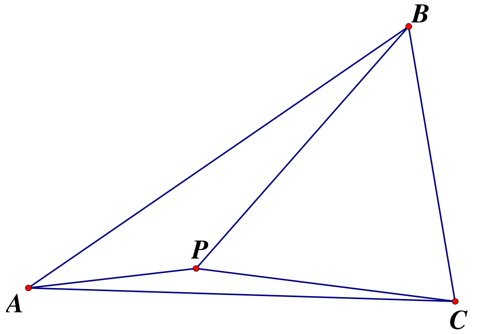
Fermat's point

Created by Mr. Francis Hung on 20161028 Last updated: 2021-09-03

Given a triangle ABC with each angle < 120°. Find a point P inside ΔABC such that $S = PA + PB + PC$ is a minimum.
Express S in terms of a, b and c.

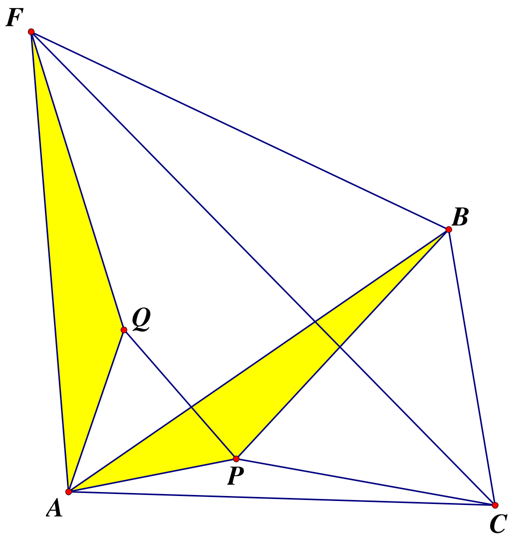


Rotate P anticlockwise by 60° about A to Q. Rotate B anticlockwise by 60° about A to F. Join PQ, QF, BF and CF.

By the property of rotation,

$AP = AQ$, $AB = AF$ and $\angle PAQ = 60°$, $\angle BAF = 60°$

ΔAPQ and ΔABF are equilateral triangles



$\angle BAP = 60° - \angle BAQ = \angle FAQ$

$\therefore \Delta ABP \cong \Delta AFQ$ (S.A.S.)

$S = PA + PB + PC$

$= PQ + FQ + PC$ (corr. sides, $\cong \Delta$s)

$\geq CF$ (the shortest distance between C and F is a straight line)

When S attains its minimum, P and Q must lie on CF.

In this case, $\angle APC = 180° - 60° = 120°$ (adj. $\angle$s on st. line) ⋯ (*)

If we construct two more equilateral triangles BCD and ACE outwards, then $S \geq AD$ and $S \geq BF$

It can be proved easily that $\Delta ABD \cong \Delta FBC$ (S.A.S.)

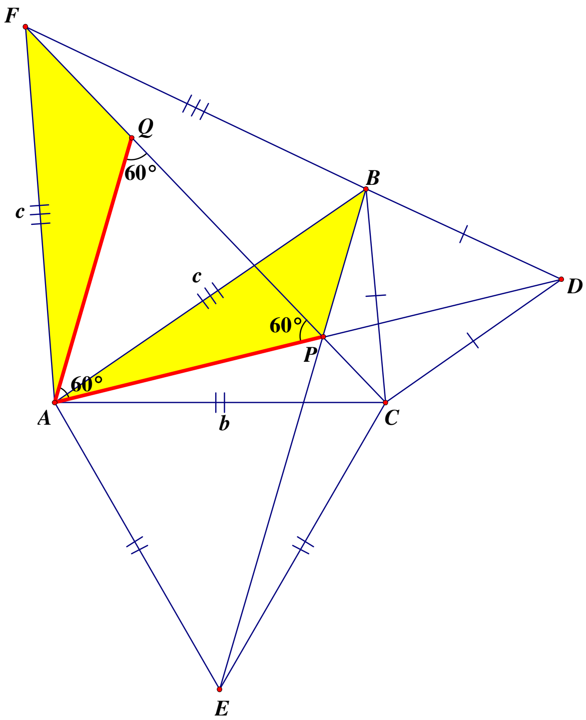
$\therefore FC = AD$ (corr. sides, $\cong \Delta$s)

Also, $\Delta BCE \cong \Delta DCA$ (S.A.S.)

$\therefore BE = DA$ (corr. sides, $\cong \Delta$s)

The minimum distance $S = AD = BE = CF$

Next, we shall prove that AD, BE and CF are concurrent at P.



Suppose AD and CF, which intersect at P. Join BP, PE.

Try to show that B, P, E are collinear.

$\angle CPD = 60°$ (vert. opp. $\angle$s)

$\angle CBD = 60°$ (property of equilateral Δ)

B, D, C, P are concyclic (converse, $\angle$s in the same segment)

$\angle BPF = \angle BDC$ (ext. $\angle$, cyclic quad.)

By the property (*), $\angle APC = 120°$

$\angle AEC = 60°$ (property of equilateral Δ)

A, P, C, E are concyclic (opp. $\angle$s supp.)

$\angle APE = \angle ACE = 60°$ ($\angle$s in the same segment)

$\angle BPF + \angle APQ + \angle APE = 60° + 60° + 60° = 180°$

B, P, E are collinear (converse, adj. $\angle$s on st. line)

AD, BE, CF concurrent at P, $\angle APC = \angle BPC = \angle APB = 120°$

In ΔACF, $CF^2 = b^2 + c^2 - 2bc\cos(A + 60°)$

$CF^2 = b^2 + c^2 - bc(\cos A - \sqrt{3}\sin A)$ (compound angle formula)

$$= b^2 + c^2 - bc\cdot\frac{b^2 + c^2 - a^2}{2bc} + \sqrt{3}bc\sin A$$

$$= \frac{a^2 + b^2 + c^2}{2} + 2\sqrt{3}\omega$$, where ω is the area of ΔABC

The shortest sum of distances from P to each vertex $= CF = \sqrt{\frac{a^2 + b^2 + c^2}{2} + 2\sqrt{3}\omega}$

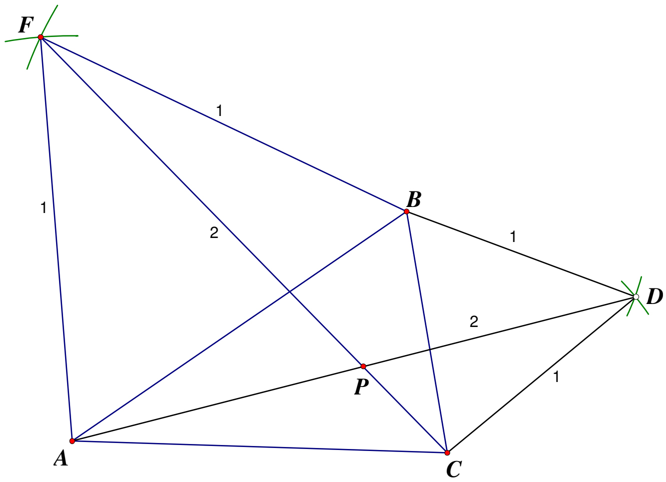
Use ruler and compasses to draw the Fermat's point of a triangle.

Suppose all interior angles of $\Delta ABC < 120°$

(1) Draw the equilateral triangles ABF and BCD outwards.

(2) Join AD and CF, which intersect at P.

Then P is the Fermat's point.



If one interior angle of ΔABC, say, $\angle B \geq 120°$, then B will be the Fermat's point.

Proof: The intersection point of P will lie outside ΔABC.

$\angle APC = 120°$ by (*)

$\angle AEC = 60°$ (property of equilateral Δ)

$\angle APC + \angle AEC = 180°$

A, P, C, E are concyclic (opp. $\angle$s supp.)

$\angle APE = \angle ACE = 60°$ ($\angle$s in the same segment)

$\angle CPE = 120° - 60° = 60°$

Let $\angle CEB = \theta$, then $0° < \theta < 60°$ ⋯⋯ (**)

$\angle CAP = \angle CEP = \theta$ ($\angle$s in the same segment)

$\angle BAP < \angle CAP = \theta$

$\angle ABP = 180° - (\angle APB + \angle BAP)$ ($\angle$ sum of Δ)

$= 180° - 60° - \angle BAP$

$= 120° - \angle BAP$

$> 120° - 60° = 60°$ by (**)

$= \angle APB$

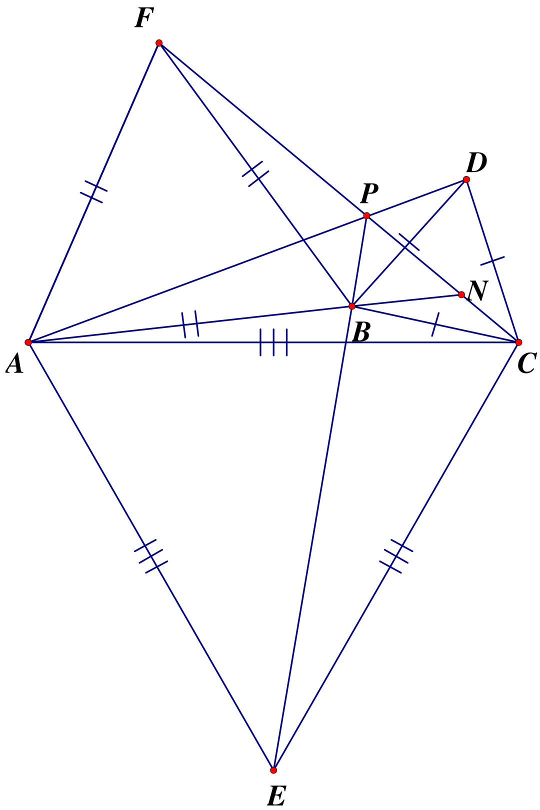
$AP > AB$ (greater sides opp. greater angles)

In a similar manner, $\angle CBP > \angle CPB = 60°$

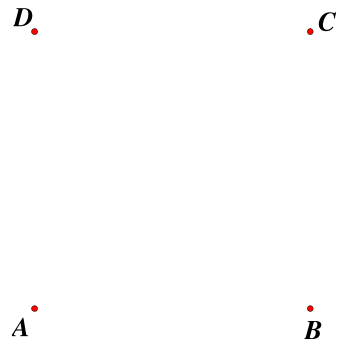
$CP > CB$ (greater sides opp. greater angles)

$S = BA + BB + BC = BA + BC < PA + PB + PC$

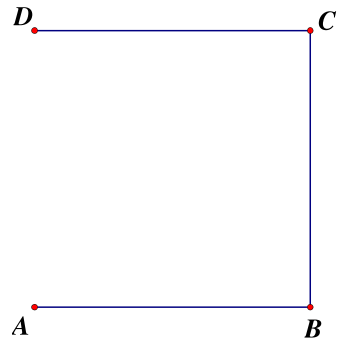
$\therefore$ B is the Fermat's point.



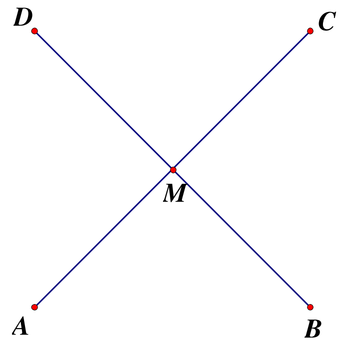
Problem: A, B, C, D are 4 points lying on the corners of a square, length = 1. Build a road network connecting A, B, C and D so that the total length of the network is shortest.



First solution:
Join AB, BC and CA. Total length = 3



Second solution:
Join the diagonals AC and BD which intersect at M.
Total length = $AC + BD = \sqrt{2} + \sqrt{2} = 2\sqrt{2} \approx 2.82$
The second solution is shorter than the first one.



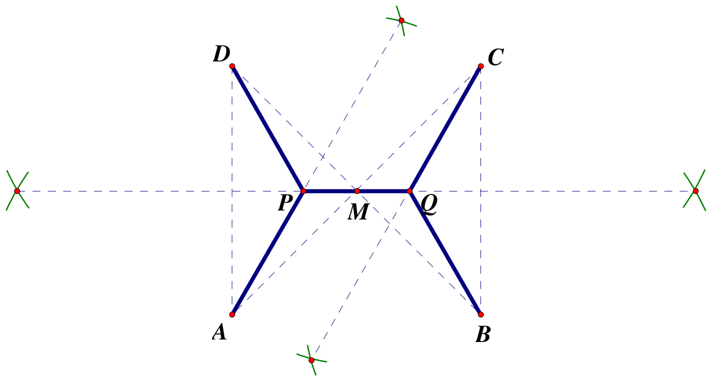
Third solution
Join AC, BD which intersect at M.
Let P and Q be the Fermat's point of ΔADM and ΔBCM respectively.
The network joining AP, PD, PQ, BQ and CQ will have the shortest distance.
Proof: By the property of Fermat's point,
$S = PA + PD + PM + QB + QC + QM$ is the least.
$PA = x$, $\angle APB = \angle BQC = 120°$, $AP = DP = BQ = CQ$
$\angle PAD = \angle PDA = 30°$

$$1 = AD = 2x \cos 30° = \sqrt{3}x \Rightarrow x = \frac{1}{\sqrt{3}}$$

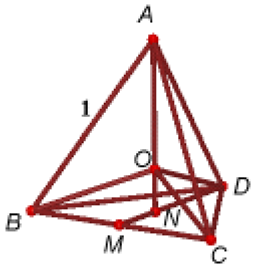
$$PQ = 1 - 2\,AP \sin 30° = 1 - x = 1 - \frac{1}{\sqrt{3}}$$

$$S = 4x + PQ = \frac{4}{\sqrt{3}} + 1 - \frac{1}{\sqrt{3}} = 1 + \frac{3}{\sqrt{3}} = 1 + \sqrt{3} \approx 2.73$$

This answer is smaller than that in the second solution.



Problem 2: *ABCD* are vertices of a regular tetrahedron, length = 1. Build a network connecting *A*, *B*, *C* and *D* so that the total length of the network is the shortest.



$AB = BC = CA = AD = CD = BD = 1$ and ΔABC, ΔACD, ΔABD, ΔBCD are equilateral.

Let M be the mid-points of BC, $BM = MC = 0.5$

Join DM, then $\Delta BDM \cong \Delta CDM$ (S.S.S.)

$\angle BMD = \angle CMD = 90°$ (corr. ∠s ≅ Δs, adj. ∠s on st. line)

$DM = \sqrt{1^2 - 0.5^2} = \dfrac{\sqrt{3}}{2}$ (Pythagoras' theorem)

DM is the median of ΔBCD

Let N be the centriod of ΔBCD

$DN = \dfrac{2}{3}\cdot\dfrac{\sqrt{3}}{2} = \dfrac{\sqrt{3}}{3}$ (property of centriod)

Join AN. Then AN is the altitude of the tetrahedron ($AN \perp \Delta BCD$)

$AN = \sqrt{AD^2 - ND^2} = \sqrt{1^2 - \left(\dfrac{\sqrt{3}}{3}\right)^2} = \dfrac{\sqrt{6}}{3}$ (Pythagoras' theorem)

Suppose O is the centre of the tetrahedron. Then $OA = OB = OC = OD = x$ and $ON = AN - AO = \dfrac{\sqrt{6}}{3} - x$

In ΔOND, $ON^2 + DN^2 = OD^2$ (Pythagoras' theorem)

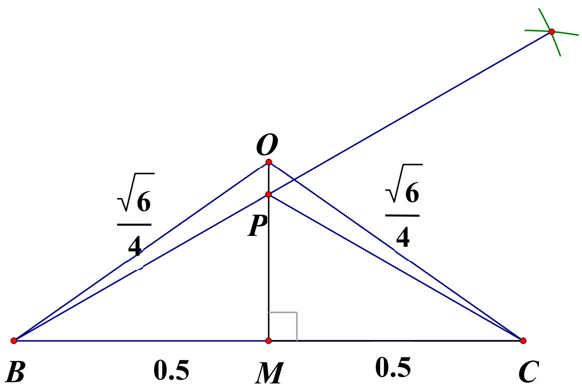
$$\left(\frac{\sqrt{6}}{3} - x\right)^2 + \left(\frac{\sqrt{3}}{3}\right)^2 = x^2$$

$$\frac{2}{3} - \frac{2\sqrt{6}}{3}x + x^2 + \frac{1}{3} = x^2$$

$$x = \frac{3}{2\sqrt{6}} = \frac{\sqrt{6}}{4}$$

AOB, BOC, COD and DOA are 4 congruent triangles dividing the tetrahedron into 4 equal parts. Let us consider ΔBOC.

If P is the Fermat's point then P lies on OM and $\angle BPC = 120°$



$OM = \sqrt{\left(\dfrac{\sqrt{6}}{4}\right)^2 - \left(\dfrac{1}{2}\right)^2} = \dfrac{\sqrt{2}}{4}$ (Pythagoras' theorem)

$\angle BPM = \angle CPM = 60°$

$PM = MC \div \tan 60° = \dfrac{1}{2} \div \sqrt{3} = \dfrac{\sqrt{3}}{6}$

$OP = OM - PM = \dfrac{\sqrt{2}}{4} - \dfrac{\sqrt{3}}{6}$

$PC^2 = PM^2 + MC^2$ (Pythagoras' theorem)

$PC = \sqrt{\left(\dfrac{\sqrt{3}}{6}\right)^2 + \left(\dfrac{1}{2}\right)^2} = \dfrac{\sqrt{3}}{3}$

So the shortest sum of distances from P to O, B, C is $\dfrac{\sqrt{2}}{4} - \dfrac{\sqrt{3}}{6} + \dfrac{2\sqrt{3}}{3} = \dfrac{\sqrt{2}}{4} + \dfrac{\sqrt{3}}{2}$.

Now, let Q be the Fermat's point of ΔAOD, then the shortest sum of distance $= \dfrac{\sqrt{2}}{4} + \dfrac{\sqrt{3}}{2}$

The shortest total length of network is $2\left(\dfrac{\sqrt{2}}{4} + \dfrac{\sqrt{3}}{2}\right) = \dfrac{\sqrt{2}}{2} + \sqrt{3} \approx 2.439 < OA + OB + OC + OD = \sqrt{6} \approx 2.449$