Napoleon triangle

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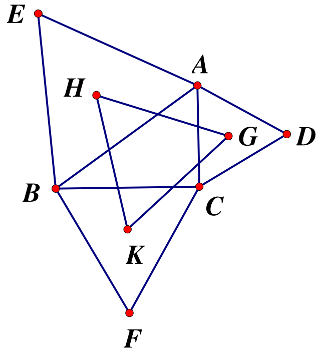
Given a triangle ABC. Equilateral triangles ABE, ACD, BCF with centres H, G, K respectively are drawn outwards as shown. To prove HKG is an equilateral triangle.



Method 1

Let $BC = a$, $AC = b$, $AB = c$

$AH = \frac{2}{3}$ median of ΔABE

$$= \frac{2}{3} \times \frac{\sqrt{3}}{2} c = \frac{c}{\sqrt{3}}$$

$$AG = \frac{b}{\sqrt{3}}$$

$\angle BAH = 30° = \angle CAG$

$\angle HAG = \angle A + 60°$

By cosine rule on ΔAHG

$$HG^2 = \left(\frac{b}{\sqrt{3}}\right)^2 + \left(\frac{c}{\sqrt{3}}\right)^2 - \frac{2bc}{\sqrt{3}}\cos(A + 60^\circ)$$

$$HG^2 = \frac{1}{3}\left[b^2 + c^2 - 2bc\left(\cos A \cdot \frac{1}{2} - \sin A \cdot \frac{\sqrt{3}}{2}\right)\right]$$

$$HG^2 = \frac{1}{3}\left(b^2 + c^2 - bc\cos A + \sqrt{3}bc\sin A\right)$$

Cosine rule on ΔABC: $a^2 = b^2 + c^2 - 2bc\cos A$

Sine rule: $\frac{a}{\sin A} = 2R$

$$HG^2 = \frac{1}{3}\left(b^2 + c^2 + \frac{a^2}{2} - \frac{b^2}{2} - \frac{c^2}{2} + \frac{\sqrt{3}abc}{2R}\right)$$

$$= \frac{1}{3}\left(\frac{a^2 + b^2 + c^2}{2} + \frac{\sqrt{3}abc}{2R}\right)$$

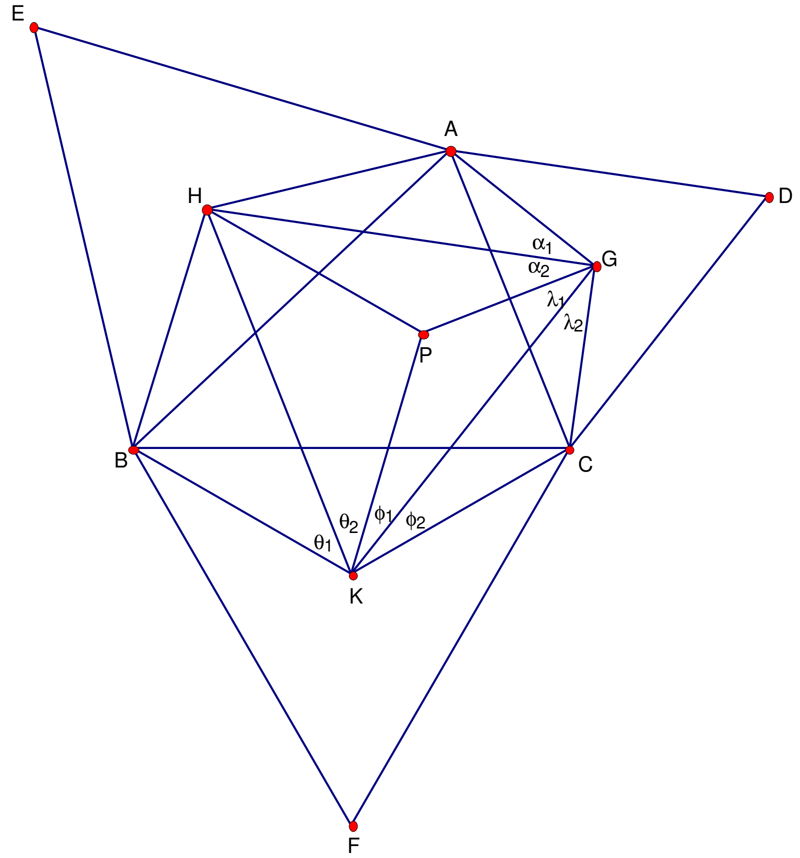
$$= \frac{1}{6}\left(a^2 + b^2 + c^2 + \frac{\sqrt{3}abc}{R}\right)$$

This is a symmetric function

Similarly $GK^2 = HK^2 = \frac{1}{6}\left(a^2 + b^2 + c^2 + \frac{\sqrt{3}abc}{R}\right)$

$\therefore$ HGK is an equilateral triangle

Method 2



Let P be the mirror image of the reflection of A about HG.

Then $\Delta AHG \cong \Delta PHG$ (S.S.S.)

$HA = HP$ (corr. sides $\cong \Delta$s)

$\because$ H = centre of ΔEAB

$\therefore$ $HA = HB = HE$

We can draw a circle with H as centre to pass through $EAPB$.

$\angle APB + \angle AEB = 180°$ (opp. $\angle$s cyclic quad.)

$\angle APB = 180° - 60° = 120°$

In a similar manner, $\angle APC = 120°$

$\angle BPC = 360° - 120° - 120° = 120°$ ($\angle$s at a pt.)

F,B,P,C are concyclic (opp. $\angle$s supp.)

$KB = KC = KF$

K is the centre of the circle $FBPC$.

$KP = KB = KC$

$\Delta BHK \cong \Delta PHK$, $\Delta CGK \cong \Delta PGK$ (S.S.S.)

Let $\theta_1,\theta_2,\phi_1,\phi_2,\alpha_1,\alpha_2,\lambda_1,\lambda_2$ be as shown.

$\theta_1 = \theta_2$, $\phi_1 = \phi_2$, $\alpha_1 = \alpha_2$, $\lambda_1 = \lambda_2$ corr. $\angle$s $\cong \Delta$s

$\because$ $\angle BKC = 120°$, $\angle AGC = 120°$

$\therefore$ $\theta_1 + \theta_2 + \phi_1 + \phi_2 = 120°$, $\alpha_1 + \alpha_2 + \lambda_1 + \lambda_2 = 120°$

$2(\theta_2+\phi_1) = 120°$, $2(\alpha_2+\lambda_1) = 120°$

$\theta_2 + \phi_1 = 60°$, $\alpha_2 + \lambda_1 = 60°$

$\angle HKG = 60°$, $\angle HGK = 60°$

$\angle KHG = 180° - 60° - 60° = 60°$ ($\angle$ sum of Δ)

ΔHKG is an equilateral Δ