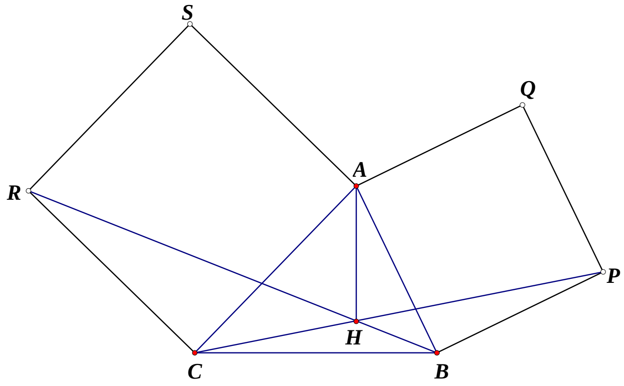
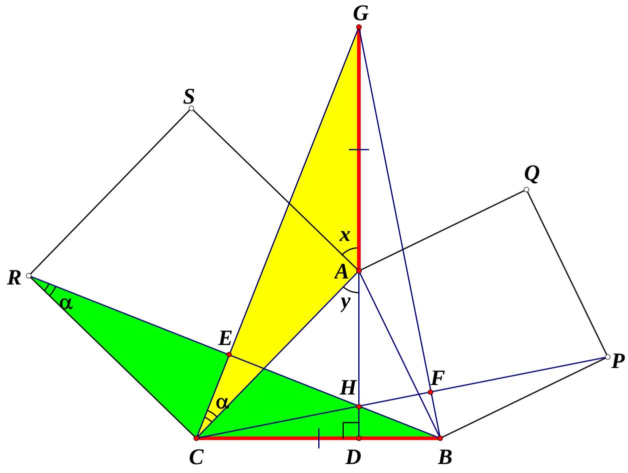
Triangle Problem 2

Created by Francis Hung Last updated: 03 September 2021

In ΔABC, squares $ABPQ$, $ACRS$ are drawn outwards as shown. The lines BR, CP intersect at H. Prove that $AH \perp BC$.





Let AD be the altitude $\perp BC$ (D lies on BC).

Produce DA to G so that $AG = BC$.

Let $\angle SAG = x$, $\angle CAD = y$.

Then $x + 90° + y = 180°$ (adj. $\angle$s on st. line)

$\Rightarrow y = 90° - x$

$\angle ACD = 90° - y$ ($\angle$ sum of ΔACD)

$= x$

$\therefore \angle CAG = 90° + x = \angle RCB$

$RC = AC$ (sides of a square)

$BC = AG$ (construction)

$\therefore \Delta AGC \cong \Delta CBR$ (S.A.S.)

In a similar manner, $\Delta AGB \cong \Delta BCP$ (S.A.S.)

$\therefore \angle ACG = \angle CRB = \alpha$ (corr. $\angle$s, $\cong \Delta$s)

Suppose CG intersects BR at E.

$\angle ECR = 90° - \alpha$

$\therefore \angle CER = 180° - (90° - \alpha) - \alpha$ ($\angle$ sum of ΔRCE)

$= 90°$

Suppose CP intersects BG at F.

In a similar manner $\angle BFP = 90°$

$\therefore GD$, BE, CF are the altitudes of ΔGBC.

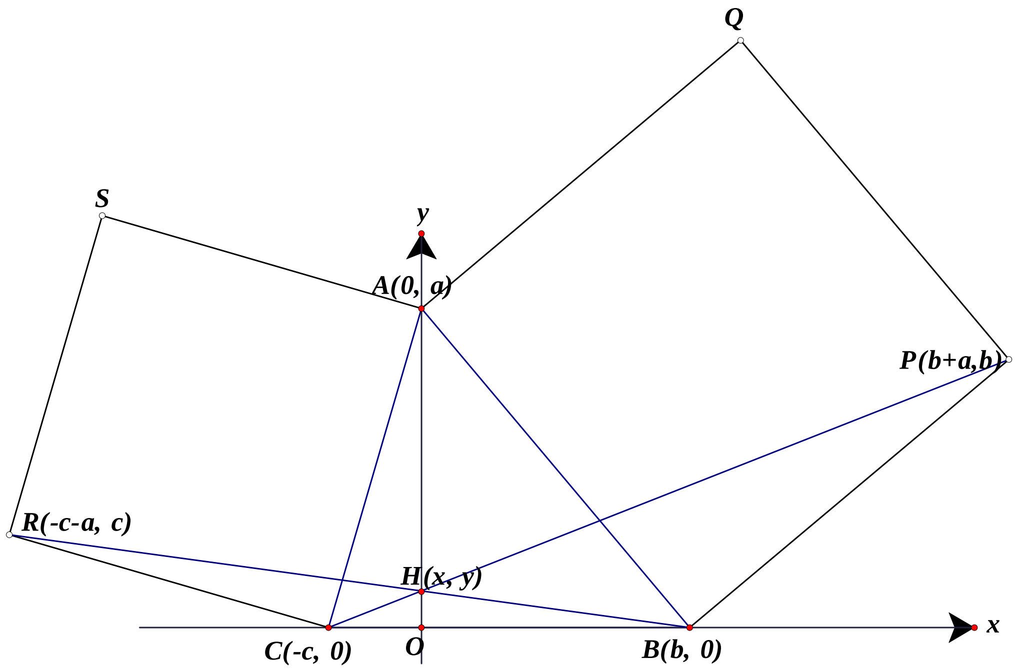
$\because$ The 3 altitudes of ΔGBC intersect at one point: orthocentre H.

$\therefore H$ lies in GAD

$\Rightarrow AH \perp BC$.

Method 2 Introduce a rectangular coordinates system as shown.

Let the coordinates of A, B and C be $(0, a)$, $(b, 0)$ and $(–c, 0)$ respectively.



Then the coordinates of P and R are $(b + a, b)$ and $(–c – a, c)$ respectively.

Suppose CP and BR intersect at H. Let the coordinates of H be (x, y).

$$m_{CH} = m_{CP} \quad \Rightarrow \quad \frac{y}{x+c} = \frac{b}{a+b+c} \Rightarrow y = \frac{b}{a+b+c}\left(x+c\right) \cdots (1)$$

$$m_{BH} = m_{RB} \quad \Rightarrow \quad \frac{y}{x-b} = \frac{c}{-c-a-b} \Rightarrow y = -\frac{c}{a+b+c}\left(x-b\right) \cdots (2)$$

$$(1) = (2): \quad \frac{b}{a+b+c}\left(x+c\right) = -\frac{c}{a+b+c}\left(x-b\right)$$

$b(x + c) = –c(x – b)$

$bx + bc = –cx + bc$

$bx + cx = 0$

$(b + c)x = 0$

$x = 0$

$\therefore H(0, y)$

i.e. $AH \perp BC$