Triangle Problem 3

Created by Francis Hung on 2011-12-8 Last updated: 03 September 2021

Theorem 1

Given a ΔABC with $AB = AC$. M is the mid-point of AC. $\angle ABM = \alpha$, $\angle CBM = \beta$, $\angle BMC = \theta$. Then

(a) if $AB = BC$, then $\alpha = \beta$.

(b) if $AB > BC$, then $\alpha < \beta$;

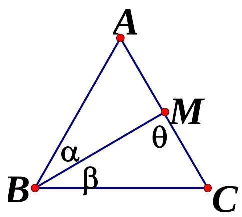
(c) if $AB < BC$, then $\alpha > \beta$.

(a) Given $AB = BC$.

ΔABC is an equilateral triangle

$\Delta ABM \cong \Delta CBM$ (S.S.S.)

$\alpha = \beta = 30°$ (corr. sides, $\cong\Delta$s)

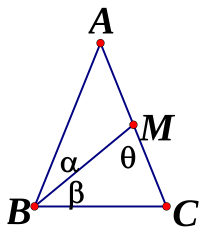


(b) Given $AB > BC$.

Apply sine rule on ΔABM and ΔBCM.

$\frac{AM}{\sin\alpha} = \frac{AB}{\sin(180^\circ - \theta)}$ ……… (1)

$\frac{CM}{\sin\beta} = \frac{BC}{\sin\theta}$ …… (2)



Using the fact that $AM = CM$ and $\sin(180° – \theta) = \sin\theta$

(1) ÷ (2): $\frac{\sin\beta}{\sin\alpha} = \frac{AB}{BC}$

$\because AB > BC \therefore \sin\beta > \sin\alpha$

$\alpha < \beta$

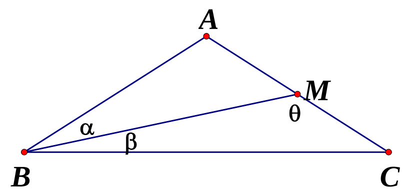
(c) Given $AB < BC$.

Using the same method as in (b), we have

$\frac{\sin\beta}{\sin\alpha} = \frac{AB}{BC}$

$\because AB < BC \therefore \sin\beta < \sin\alpha$

$\alpha > \beta$



Theorem 2 Given a ΔABC with $AB = AC$. M is the mid-point of AC. G is the centriod of ΔABC. The circumscribed circle BGC is drawn. Let $AM = x = MC$. Then

(a) if $AB = BC$, then AC touches the circle at C, $\angle CBM = \alpha$ and $AC = BC$;

(b) if $AB > BC$, then AC cuts the circle at P, $\angle PBM = \alpha$ and $PA = PB + PC$;

(c) if $AB < BC$, then AC produced cuts the circle at P, $\angle PBM = \alpha$ and $PA + PC = PB$.

(a) Given $AB = BC$.

ΔABC is an equilateral triangle

Clearly $AC = BC$.

Let N be the mid-point of AB.

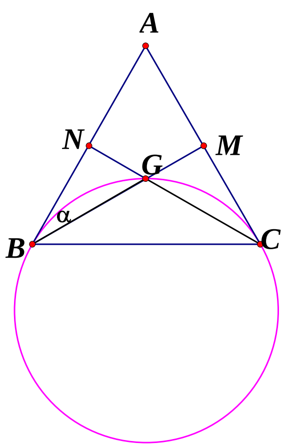
$\Delta ACN \cong \Delta BCN$ (S.S.S.)

$\angle ACN = \angle BCN = 30°$ (corr. sides, $\cong\Delta$s)

By theorem 1 (a), $\angle CBM = 30°$

$\therefore \angle ACN = \angle CBM$

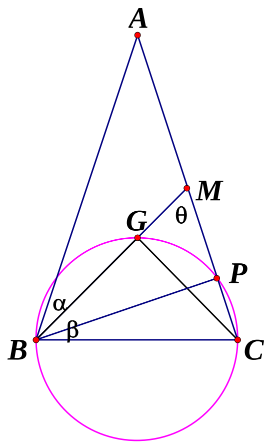
$\therefore AC$ touches the circle at C (converse, $\angle$ in alt. seg.)



(b) Given $AB > BC$.

By theorem 1 (b), $\alpha < \beta$

We can find a point P between CM so that $\angle PBM = \alpha$.



$\because AB = AC$ and G is the centriod

It is easy to show that $BG = CG$ and $\angle ABG = \alpha = \angle ACG$

$\therefore \angle PCG = \alpha = \angle PBG$

$\therefore BCPG$ is a cyclic quadrilateral (converse, $\angle$s in the same seg.)

$\therefore P$ lies on the circle.

Apply sine law on ΔABM and ΔBPM.

$$\frac{AM}{AB} = \frac{\sin\alpha}{\sin(180^\circ - \theta)} \quad \cdots\cdots (3)$$

$$\frac{PM}{BP} = \frac{\sin\alpha}{\sin\theta} \quad \cdots\cdots (4)$$

Using the fact that $\sin(180° – \theta) = \sin \theta$ and $AM = x = MC$, $AB = 2x$.

(3) = (4): $\frac{AM}{AB} = \frac{PM}{BP} = \frac{1}{2}$

$PM = MC – PC = x – PC$

$PA = PM + AM = 2x – PC$

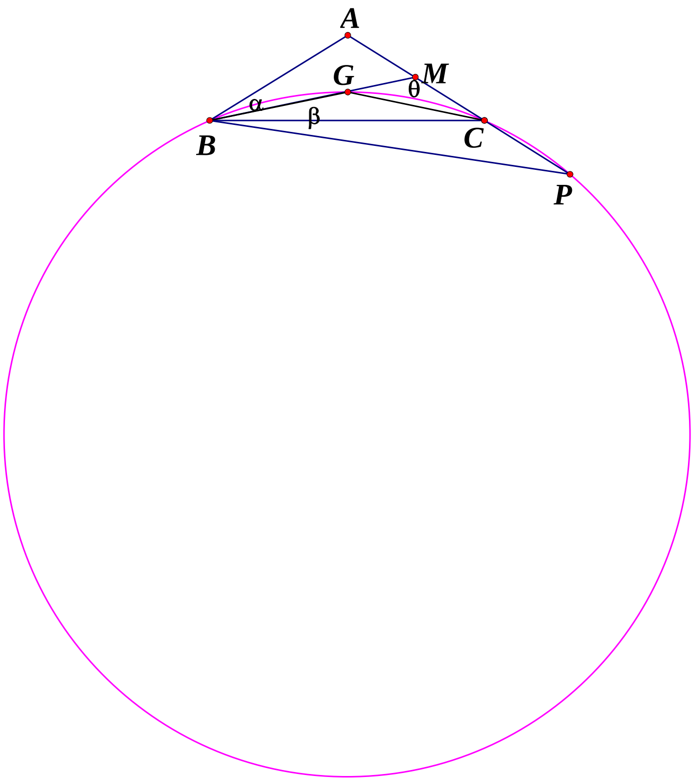
$PB = 2PM = 2x – 2PC$

$PA = 2x – PC = 2x – 2PC + PC = PB + PC$

(c) Given $AB < BC$.

By theorem 1 (c), $\alpha > \beta$

We can find a point P on MC produced so that $\angle PBM = \alpha$.



$\because AB = AC$ and G is the centriod

It is easy to show that $BG = CG$ and $\angle ABG = \alpha = \angle ACG$

$\therefore \angle ACG = \alpha = \angle PBG$

$\therefore BPCG$ is a cyclic quadrilateral (ext. $\angle$ = int. opp. $\angle$)

$\therefore P$ lies on the circle.

Apply sine law on ΔABM and ΔBPM.

$$\frac{AM}{AB} = \frac{\sin\alpha}{\sin(180^\circ - \theta)} \quad \cdots\cdots (3)$$

$$\frac{PM}{BP} = \frac{\sin\alpha}{\sin\theta} \quad \cdots\cdots (4)$$

Use the fact that $\sin(180° – \theta) = \sin \theta$ and $AM = x = MC$, $AB = 2x$.

(3) = (4): $\frac{AM}{AB} = \frac{PM}{BP} = \frac{1}{2}$

$MP = MC + PC = x + PC$

$PA = AM + MP = 2x + PC$

$PB = 2PM = 2x + 2PC$

$PA + PC = 2x + PC + PC = PB$