Examples on Trinagle inequality

Reference: 1957 HKU O Level Pure Mathematics Paper 2 Q1

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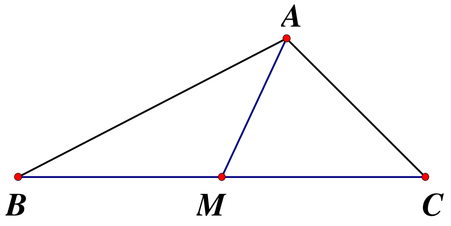
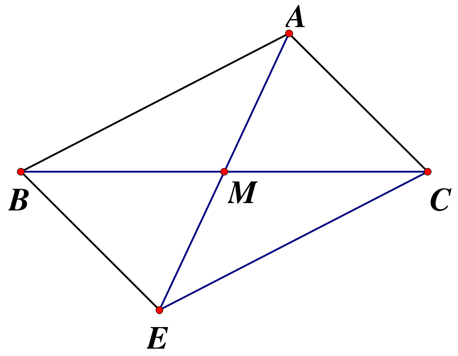
Example 1 1957 HKU O Level Pure Mathematics Paper 2 Q1(a)

(a) If M is the mid-point of the side BC of the triangle ABC,
prove that $AB + AC > 2AM$.

(b) If M is a point of the side BC of the triangle ABC such that $BM : MC = m : n$,
prove that $\frac{nAB+mAC}{m+n} > AM$.

(a) Produce AM to E such that $AM = ME$.





$BM = MC$	(given)
$AM = ME$	(By construction)
$ABEC$ is a //-gram	(diagonal bisect each other)
$\therefore AB = CE$	(opp. sides //-gram)
In ΔACE, $AC + CE > AE$	(triangle inequality)
$\therefore AB + AC > 2AM$	

(b) $\overrightarrow{AM} = \frac{n\overrightarrow{AB}+m\overrightarrow{AC}}{m+n}$

$$AM^2 = \left|\overrightarrow{AM}\right| \cdot \left|\overrightarrow{AM}\right| = \left|\frac{n\overrightarrow{AB}+m\overrightarrow{AC}}{m+n}\right| \cdot \left|\frac{n\overrightarrow{AB}+m\overrightarrow{AC}}{m+n}\right|$$

$$(m+n)^2 AM^2 = n^2\left|\overrightarrow{AB}\right|^2 + 2nm\overrightarrow{AB}\cdot\overrightarrow{AC} + m^2\left|\overrightarrow{AC}\right|^2$$

$$= n^2 AB^2 + 2nm(AB)(AC)\cos\angle BAC + m^2AC^2$$

$$\leq n^2 AB^2 + 2nm(AB)(AC) + m^2AC^2$$, (equality holds when $\angle BAC = 0°$)

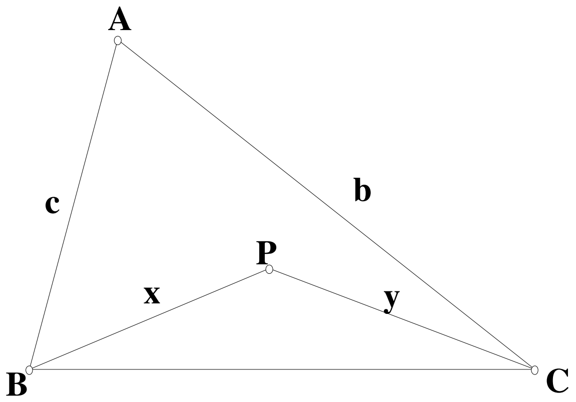
$$= (n\,AB + m\,AC)^2$$

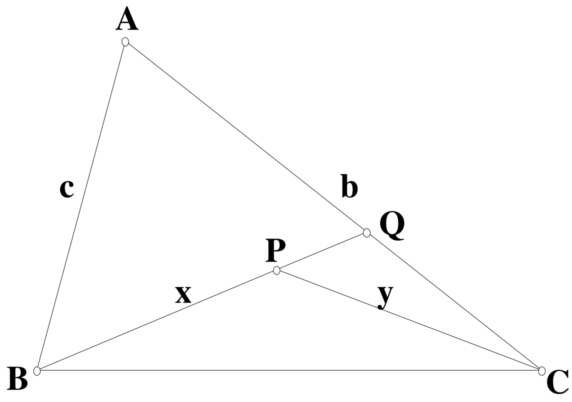
$\therefore \frac{nAB+mAC}{m+n} > AM$ for $\angle BAC \neq 0°$

Example 2

In ΔABC, $AB = c$, $AC = b$, P is a point <u>inside</u> ΔABC. $BP = x$, $CP = y$. prove that $b + c > x + y$.

Proof:





Product BP to Q on AC

In ΔABQ, $c + AQ > x + PQ$ (Δ inequality) ⋯⋯(1)

In ΔCPQ, $PQ + QC > y$ (Δ inequality) ⋯⋯ (2)

(1) + (2) $PQ + (AQ + QC) + c > x + y + PQ$

$\therefore b + c > x + y$

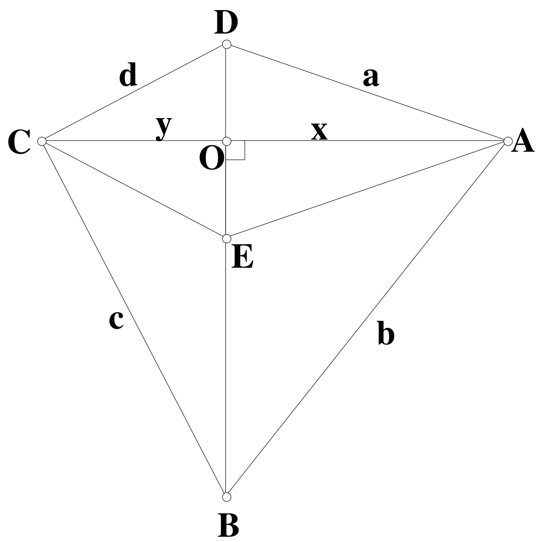
Example 3

$ABCD$ is a convex quadrilateral such that the diagonals are perpendicular which intersects at O and $OA > OC$ and $OB > OD$. To prove $AD + BC > AB + CD$.

Let the letters a, b, c, d, x, y be as shown.

Reflect ΔACD along the line AC to ΔACE.

$\because OD < OB$ $\therefore OB > OE \Rightarrow E$ lies <u>inside</u> ΔABC.



By **Example 2**, $b + c > a + d$ ⋯⋯ (1)

Apply Pythagoras' Theorem on ΔAEO, ΔCEO, ΔABO, ΔCBO

$OD^2 = a^2 - x^2 = d^2 - y^2 \Rightarrow x^2 - y^2 = a^2 - d^2$

$OB^2 = b^2 - x^2 = c^2 - y^2 \Rightarrow x^2 - y^2 = b^2 - c^2$

$x^2 - y^2 = a^2 - d^2 = b^2 - c^2$

$(a + d)(a - d) = (b + c)(b - c)$

$\frac{b+c}{a+d} = \frac{a-d}{b-c}$ ⋯⋯ (2)

In (1) $\frac{b+c}{a+d} > 1$

$\Rightarrow$ (2) $\frac{a-d}{b-c} > 1$

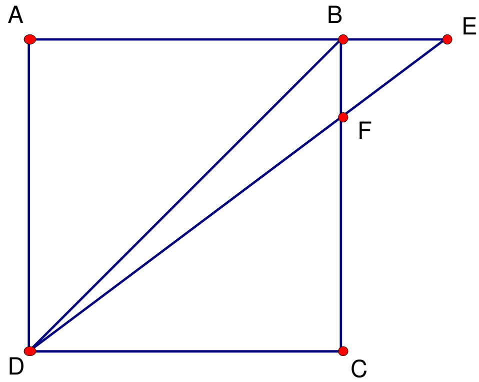
$a - d > b - c$

$a + c > b + d$

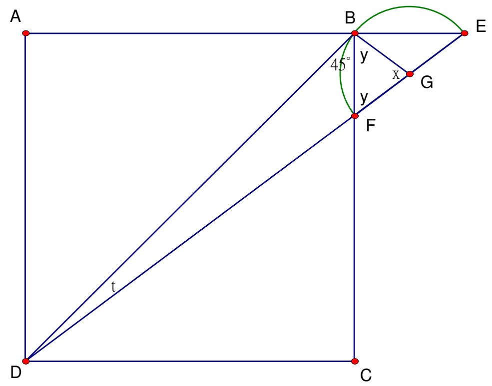
$AD + BC > AB + CD$. The result is proved.

Example 4 1971 普通數學課程一試卷二 Q11

In the figure, $ABCD$ is a square, ABE and DFE are straight lines. Prove that $DE + DF > 2BD$.



4. Let G be the mid point of EF, $\angle BDG = t$, $\angle FBG = y$, $\angle BGF = x$.



Then $\angle DBG = 45° + y$

Use G as centre, FG as radius to draw a semi-circle.

B lies on the semi-circle (converse, $\angle$ in semi-circle)

$BG = FG$ (radii)

$\angle GFB = y$ (base $\angle$s, isos. Δ)

$x = 180° - 2y$ ($\angle$s sum of Δ) ....(1)

$y = 45° + t$ (ext. $\angle$ of ΔBDF)

$y > 45°$

$3y > 135°$

$45° + y > 180° - 2y$

$45° + y > x$ (by (1))

$\angle DBG > \angle BGD$

$DG > BD$ (bigger side opp. bigger $\angle$)

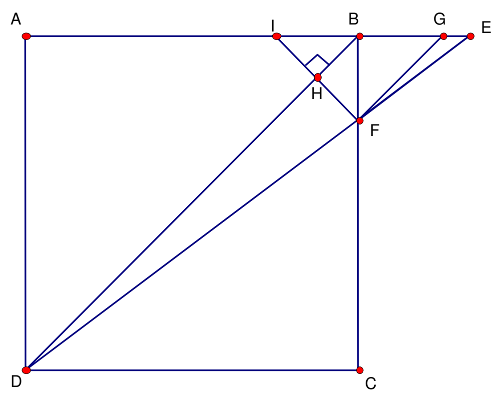
$\frac{1}{2}(DE + DF) > BD$

$DE + DF > 2BD$

Method 2

Draw a line $FG \parallel DB$, cutting AE at G.

Draw a line $FI \perp FG$, cutting DB at H.



Then it can be easily shown that

$\Delta IBH \cong \Delta BHF$ (ASA)

$\therefore IH = FH$ (corr. sides $\cong\Delta$'s)

$IB = BG$ (intercept theorem)

$FG = 2BH$ (mid pt. theorem) ....(1)

In ΔDHF, $DF > DH$ ....(2)

In ΔEFG, $EF > FG$ ....(3)

$DF + DE = DF + DF + EF = 2DF + EF$

$> 2DH + FG$ (by (2) and (3))

$= 2DH + 2BH = 2BD$ (by (1))

Example 5 1957 HKU O Level Pure Mathematics Paper 2 Q1b

ABCD is a quadrilateral. *M*, *N* are the mid-points of the opposite sides *AB*, *CD*.

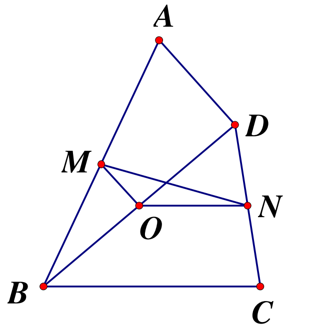
Prove that $AD + BC \geq 2MN$. If $AD + BC = 2MN$, show that *AD* is parallel to *BC*.

Join *BD*. Let *O* the mid-point of *BD*. Join *OM*, *ON*.

$OM + ON \geq MN$ (triangle inequality)

$AD = 2OM$, $BC = 2ON$ (mid-point theorem)

$\therefore AD + BC = 2OM + 2ON \geq 2MN$



To prove the second part. Suppose $AD + BC = 2MN$

Then $OM + ON = MN$

M, *O*, *N* are collinear.

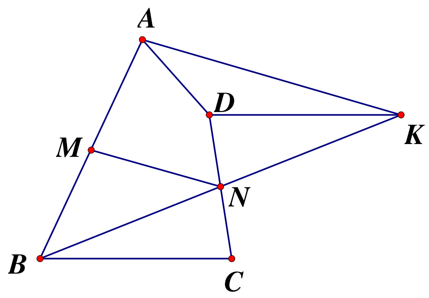
$AD \parallel MO \parallel MN$, $BC \parallel ON \parallel MN$ (mid-point theorem)

$\therefore AD \parallel BC$ (transitive property of parallel lines)

Method 2

Produce *BN* to *K* so that $BN = NK$.

$CN = ND$	(*N* = mid-point)
$\angle BNC = \angle DNK$	(vert. opp. $\angle$s)
$BN = NK$	(by construction)
$\therefore \Delta BNC \cong \Delta DNK$	(SAS)



$AD + DK \geq AK$	(*ADK* triangle inequality)
$AD + BC \geq 2MN$	(mid-point theorem)

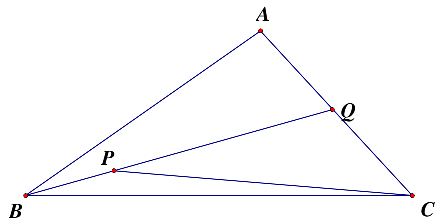
Example 6 1965 Paper II Q11

If P is any point inside a triangle ABC, prove that

$BC + CA + AB > PA + PB + PC > \frac{1}{2}(BC + CA + AB)$.

1965 Paper II Q11

Theorem Let P be any point inside ΔABC. Then $AB + AC > PB + PC$.



Proof: Join BP and produce it to cut AC at Q.

In ΔABQ, $AB + AQ > BP + PQ$ ……… (1) (Δ inequality)

In ΔCPQ, $PQ + QC > PC$ ……… (2) (Δ inequality)

(1) + (2) $AB + (AQ + QC) + PQ > BP + PC + PQ$

$\therefore AB + AC > PB + PC$ ……… (3)

The theorem is proved.

In a similar manner, $BA + BC > PA + PC$ ……… (4)

and $CA + CB > PA + PB$ ……… (5)

(3) + (4) + (5): $2(AB + BC + CA) > 2(PA + PB + PC)$

$\Rightarrow BC + CA + AB > PA + PB + PC$ ……… (*)

Next, in ΔABP, $PA + PB > AB$ ……… (6) (Δ inequality)

In ΔBCP, $PB + PC > BC$ ……… (7) (Δ inequality)

In ΔACP, $PA + PC > AC$ ……… (8) (Δ inequality)

(7) + (8) + (9): $2(PA + PB + PC) > AB + BC + CA$

$\therefore PA + PB + PC > \frac{1}{2}(BC + CA + AB)$ ……… (**)

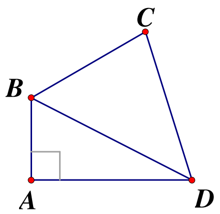
Combine (*) and (**),$BC + CA + AB > PA + PB + PC > \frac{1}{2}(BC + CA + AB)$.

Example 7 一九六七年中文中學會考乙組數學試卷二第二題

在四邊形 $ABCD$ 中，如$\angle A$ 為一直角，求證 $AB < (CD + BC)$。

一九六七年中文中學會考乙組數學試卷二第二題

在四邊形 $ABCD$ 中，如$\angle A$ 為一直角，求證 $AB < (CD + BC)$。



In ΔBCD, $CD + BC > BD$ (triangle inequality)

In ΔABD, $BD = \sqrt{AB^2 + AD^2}$ (Pythagoras' theorem)

$> AB$

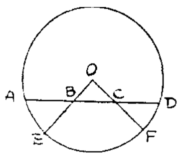
$\therefore AB < (CD + BC)$

Example 8 1969 Syllabus A Paper 3 Q16

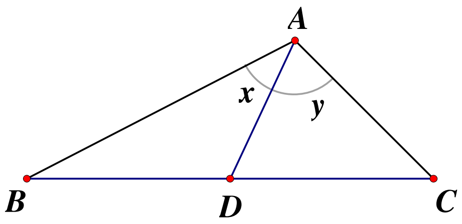
Reference: 1957 HKU O level Pure Mathematics Paper 2 Q1

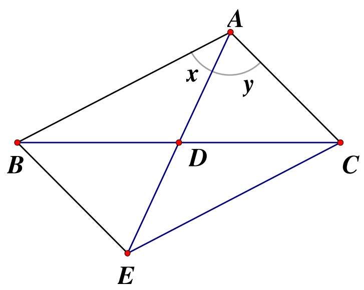
(a) In ΔABC, $AB > AC$ and AD is a median. By producing AD to E such that $AD = DE$, prove that $\angle CAD > \angle BAD$.

(b) In Figure 8, AD is a chord of circle O. $AB = BC = CD$. Using the result of (a), or by any other method, prove that arc EF > arc AE.



1969 Syllabus A Paper 3 Q16



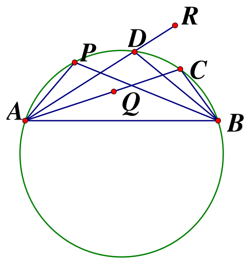


(a)	$BD = DC$	(Definition of median AD)
	$AD = DE$	(By construction)
	$ABEC$ is a //-gram	(diagonal bisect each other)
	$\therefore AB = CE$	(opp. sides //-gram)
	$\because AB > AC \therefore CE > CA$	
	$\angle CAE > \angle AEC$	(greater sides, greater $\angle$s)
	$\because \angle AEC = \angle BAD$	(alt. $\angle$s, AB // CE)
	$\therefore \angle CAD > \angle BAD$	
(b)	In ΔOAC, radius = $OA > OC$	(C lies inside the circle)
	Also, $AB = BC$	(given)
	$\therefore \angle BOC > \angle AOE$	By the result of (a)
	arc EF > arc AE	(eq. $\angle$s eq. arcs)

Example 9 1971 Syllabus A Paper 3 Q15 (*a*)

APB is a circle. Q is a point inside the circle and R is a point outside the circle. P, Q and R are on the same side of AB. Prove that $\angle AQB > \angle APB > \angle ARB$.

1971 Syllabus A Paper 3 Q15 (*a*)



Produce AQ to cut the circle at C. AR cut the circle at D.

Then $\angle APB = \angle ADB = \angle ACB$ ($\angle$s in the same seg.)

$\because AR > AD$

$\therefore \angle ABR > \angle ABD$ (greater sides opp. greater $\angle$s)

$\angle ARB = 180° - \angle DAB - \angle ABR$ ($\angle$s sum of ΔABR)

$< 180° - \angle DAB - \angle ABD$

$= \angle ADB$ ($\angle$s sum of ΔABD)

$\therefore \angle ARB < \angle ADB = \angle APB$

$\because AC > AQ$

$\therefore \angle ABC > \angle ABQ$ (greater sides opp. greater $\angle$s)

$\angle AQB = 180° - \angle QAB - \angle ABQ$ ($\angle$s sum of ΔABQ)

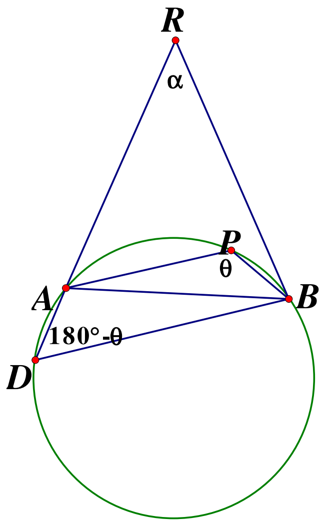
$> 180° - \angle QAB - \angle ABC$

$= \angle ACB$ ($\angle$s sum of ΔABC)

$\therefore \angle AQB > \angle ACB = \angle APB$

$\therefore \angle AQB > \angle APB > \angle ARB$

If the line segment AR does not cut the circle, let RA produced to cut the circle at D.



Let $\angle APB = \theta$, $\angle ARB = \alpha$. Join BD.

$\angle ADB = 180° - \theta$ (opp. $\angle$s cyclic quad.)

$\angle DBR = 180° - (180° - \theta) - \alpha$ ($\angle$s sum of Δ)

$= \theta - \alpha$

$\because \angle DBR = \theta - \alpha > 0$

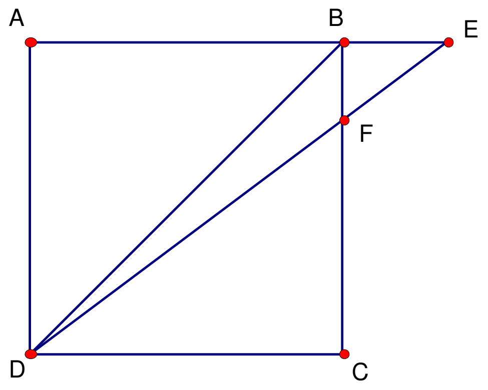
$\therefore \theta > \alpha$

$\Rightarrow \angle APB > \angle ARB$

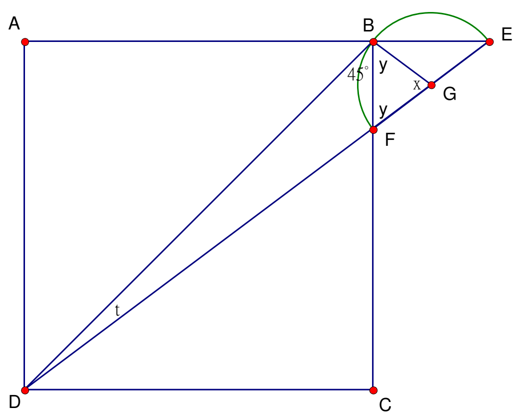
1971 普通數學課程一試卷二 Q11

在圖八中 $ABCD$ 為一正方形。ABE、DFE 均為直線。

求證 $DB < \frac{1}{2}(DE + DF)$。



Let G be the mid point of EF, $\angle BDG = t$, $\angle FBG = y$, $\angle BGF = x$.
Then $\angle DBG = 45° + y$
Use G as centre, FG as radius to draw a semi-circle.
B lies on the semi-circle (converse, $\angle$ in semi-circle)
$BG = FG$ (radii)
$\angle GFB = y$ (base $\angle$s, isos. Δ)
$x = 180° - 2y$ ($\angle$ sum of Δ) …… (1)
$y = 45° + t$ (ext. $\angle$ of ΔBDF)
$y > 45°$
$3y > 135°$
$45° + y > 180° - 2y$
$45° + y > x$ (by (1))
$\angle DBG > \angle BGD$
$DG > BD$ (bigger side opp. bigger $\angle$)
$\frac{1}{2}(DE + DF) > BD$



Method 2

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Draw a line $FI \perp FG$, cutting DB at H.
Then it can be easily shown that
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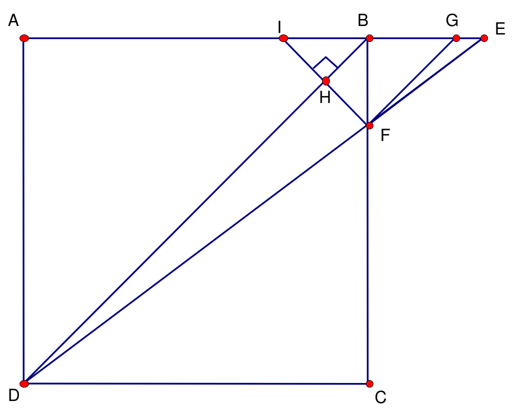
$\therefore IH = FH$ (corr. sides $\cong\Delta$s)
$IB = BG$ (intercept theorem)
$FG = 2BH$ (mid pt. theorem) …… (1)
In ΔDHF, $DF > DH$ …… (2)
In ΔEFG, $EF > FG$ …… (3)

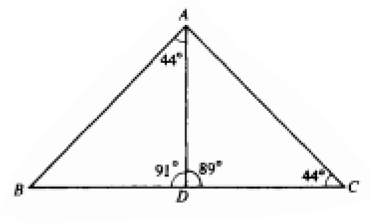
$DF + DE = DF + DF + EF = 2DF + EF$
$> 2DH + FG$ (by (2) and (3))
$= 2DH + 2BH = 2BD$ (by (1))

$\frac{1}{2}(DE + DF) > BD$



1973 香港中文中學會考普通數學課程一試卷二 Q6

圖二所示之各綫段及角度並非依正確比例或度數繪成。在綫段 AB、AD、AC、BD、DC 中，那一條最長，那一條最短？



$\angle ABD = 180° - 91° - 44° = 45°$ ($\angle$ sum of ΔABD)

$\angle BAD < \angle ABD < \angle ADB$

$BD < AD < AB$ …… (1) (greater sides opp. greater $\angle$s)

$\angle CAD = 180° - 89° - 44° = 47°$ ($\angle$ sum of ΔACD)

$\angle ACD < \angle CAD < \angle ADC$

$AD < CD < AC$ …… (2) (greater sides opp. greater $\angle$s)

In ΔABC, $\angle BAC = 44° + 47° = 91°$

$\angle ACB < \angle ABC < \angle BAC$

$AB < AC < BC$ …… (3) (greater sides opp. greater $\angle$s)

Combine (1), (2) and (3):

$BD < AD < AB$, $CD < AC < BC$

The longest side = AC, the shortest side = BD .