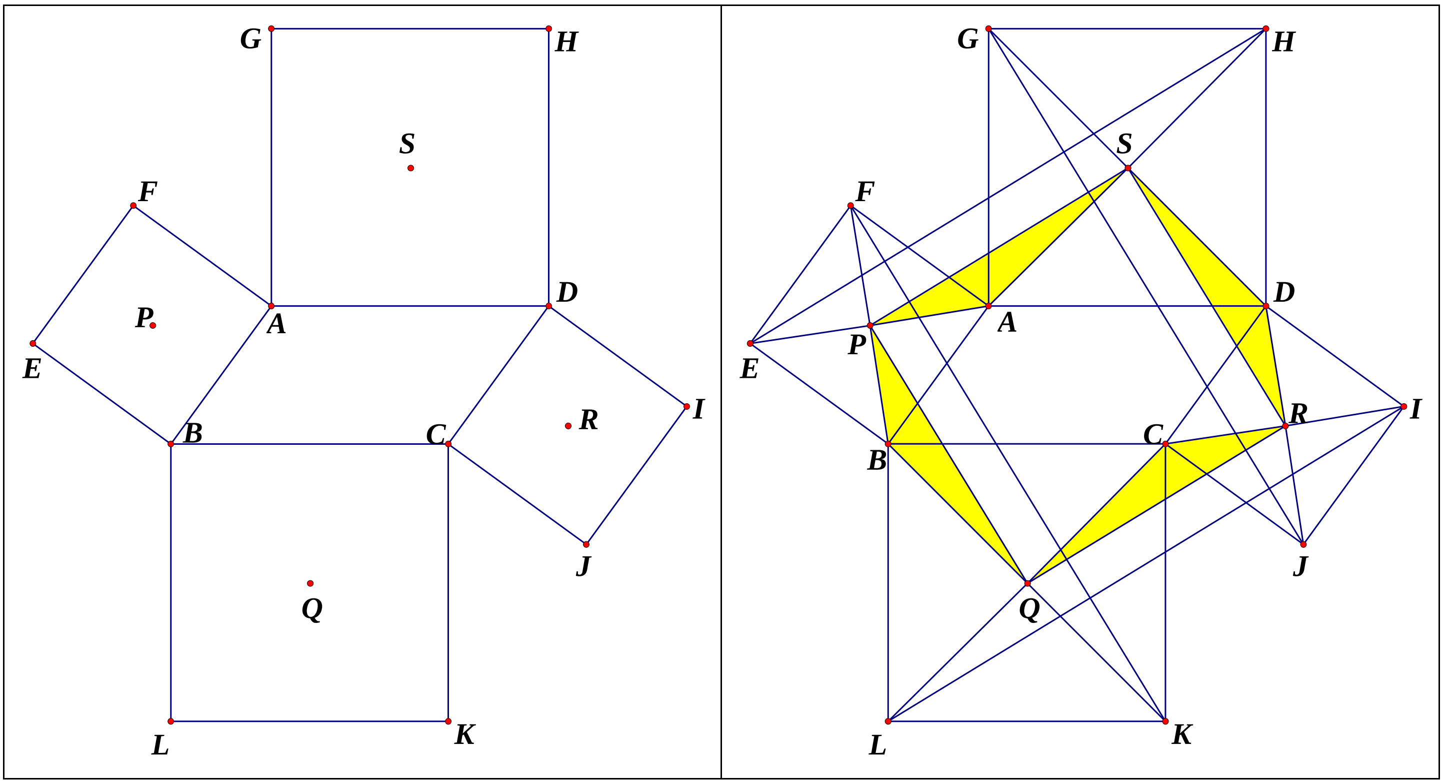
Parallelogram Square

Created by Mr. Francis Hung on 20210902 Last updated: 2022-11-14

Pearson Longman Mathematics in Action (2nd edition) Book 3A Chapter 4 P.4.24 Interesting Mathematics



In the figure, *ABCD* is a parallelogram. *ABEF*, *ADHG*, *CDIJ*, *BCKL* are squares drawn outwards.

P, *Q*, *R* and *S* are the centres of *ABEF*, *BCKL*, *CDIJ* and *ADHG* respectively. Prove that *PQRS* is a square.

Join *APE*, *BPF*, *BQK*, *CQL*, *CRI*, *DRJ*, *ASH*, *DSG*, *PQ*, *QR*, *RS*, *SP*.

$AB = CD$ and $AD = BC$ (opp. sides //-gram)

$AB = BE = EF = FA = CD = DI = IJ = JC$, $BC = CK = KL = LB = AD = DH = HG = GA$ (property of squares)

$AP = BP = EP = FP = CR = DR = JR = IR$, $BQ = CQ = KQ = LQ = AS = DS = HS = GS$ (diagonals of squares)

$\angle PBA = \angle QBC = \angle QCK = \angle RCJ = \angle RDC = \angle SDA = \angle SAG = \angle FAP = 45°$ (property of squares)

Let $\angle ABC = x = \angle ADC$ (opp. $\angle$s of //-gram), $\angle BAD = \angle BCD = 180° - x$ (int. $\angle$s, AD // BC)

$\angle BAF = \angle GAD = \angle ADH = \angle CDI = \angle DCJ = \angle BCK = \angle CBL = \angle ABE = 90°$ (property of squares)

At *A*, $\angle FAG + 90° + 180° - x + 90° = 360°$ ($\angle$s at a point) $\Rightarrow \angle FAG = x$

At *C*, $\angle KCJ + 90° + 180° - x + 90° = 360°$ ($\angle$s at a point) $\Rightarrow \angle KCJ = x$

$\therefore\ \angle PAS = 90° + x = \angle RDS = \angle RCQ = \angle PBQ$

$\Delta PAS \cong \Delta RDS \cong \Delta RCQ \cong \Delta PBQ$ (S.A.S.)

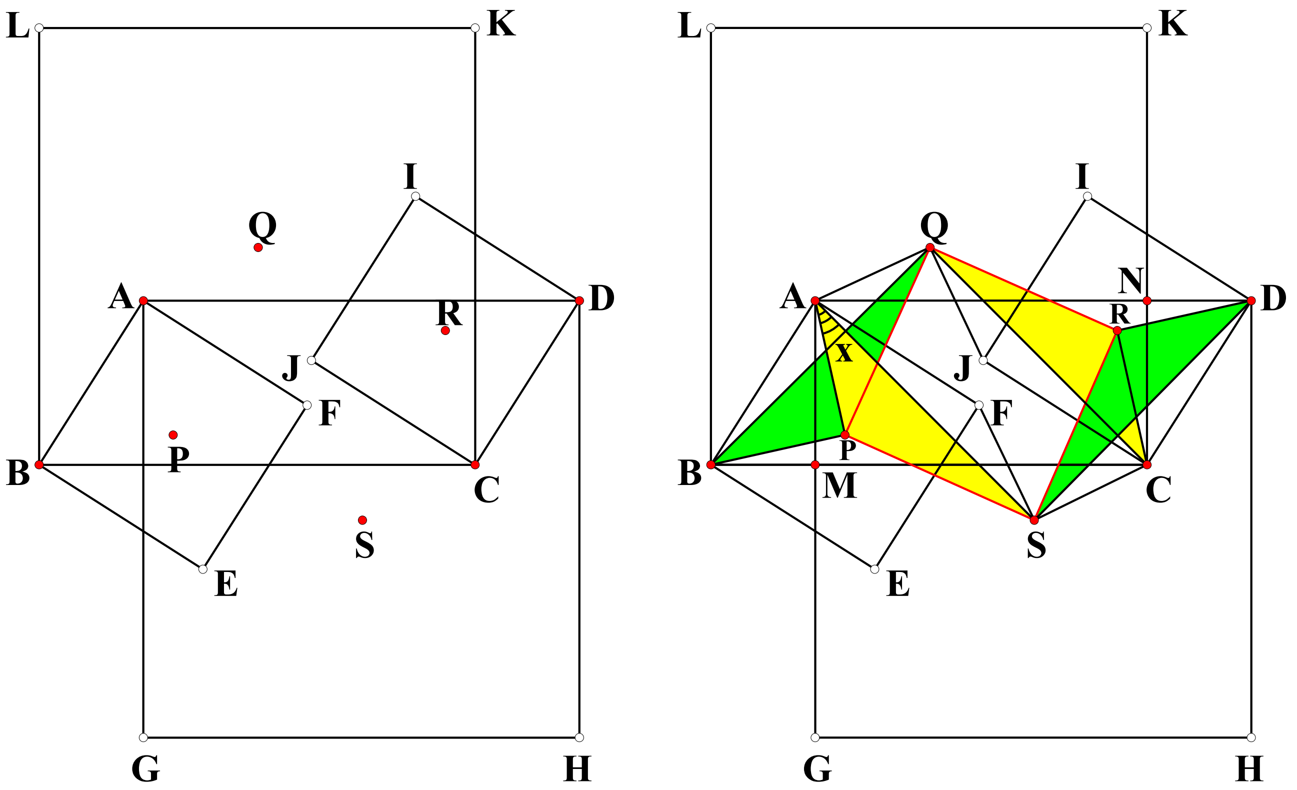
$PQ = QR = RS = SP$ (corr. sides, $\cong \Delta$s)

Let $\angle BQP = y = \angle CQR = \angle DSR = \angle ASP$ (corr. $\angle$s, $\cong \Delta$s)

$\angle APB = \angle BQC = \angle CRD = \angle DSA = 90°$ (property of squares)

$\angle PQR = 90° - y + y = 90° = \angle QRS = \angle RSP = \angle SPQ$

$\therefore$ *PQRS* is a square



In the figure, $ABCD$ is a parallelogram. $ABEF$, $ADHG$, $CDIJ$, $BCKL$ are squares drawn inwards.
P, Q, R and S are the centres of $ABEF$, $BCKL$, $CDIJ$ and $ADHG$ respectively. Prove that $PQRS$ is a square.

Join AP, BP, BQ, CQ, CR, DR, AS, DS, PQ, QR, RS, SP.
$AB = DC$ and $AD = BC$ (opp. sides //-gram)
$AB = BE = EF = FA = CD = DI = IJ = JC$, $BC = CK = KL = LB = AD = DH = HG = GA$ (property of squares)
$AP = BP = CR = DR$, $BQ = CQ = AS = DS$ (diagonals of squares)
$\angle PBA = \angle QBC = \angle QCK = \angle RCJ = \angle RDC = \angle SDA = \angle SAG = \angle FAP = 45°$ (property of squares)
Let $\angle PAS = x$
$\angle PAG = 45° - x$, $\angle BAG = 45° - (45° - x) = x$
$AD \perp AG$ (property of a square) and AD // BC (property of a //-gram)
$\therefore$ $BC \perp AG$, i.e. $\angle AMB = 90°$
$\angle ABM = 90° - x$ ($\angle$ sum of ΔABM)
$\angle ABQ = 90° - x - 45° = 45° - x = \angle CBP$
$\angle PBQ = \angle ABC - \angle ABQ - \angle CBP = 90° - x - (45° - x) - (45° - x) = x$
$\therefore$ $\angle PAS = \angle PBQ = x$
$\angle BAD = 45° + 45° + x = 90° + x = \angle BCD$ (opp. $\angle$s of //-gram)
$\angle QCR = 90° + x - 45° - 45° = x$
$\angle RCK = 45° - x$, $\angle DCK = 45° - (45° - x) = x$
$BC \perp CK$ (property of a square) and AD // BC (property of a //-gram)
$\therefore$ $AD \perp CK$, i.e. $\angle CND = 90°$
$\angle CDN = 90° - x$ ($\angle$ sum of ΔCDN)
$\angle CDS = 90° - x - 45° = 45° - x = \angle ADR$
$\angle RDS = \angle ADC - \angle ADR - \angle CDS = 90° - x - (45° - x) - (45° - x) = x$
$\therefore$ $\angle QCR = \angle RDS = x$
$\Delta PAS \cong \Delta RDS \cong \Delta RCQ \cong \Delta PBQ$ (S.A.S.)
$PQ = QR = RS = SP$ (corr. sides, $\cong$ Δs)
Let $\angle BQP = y = \angle CQR = \angle DSR = \angle ASP$ (corr. $\angle$s, $\cong$ Δs)
$\angle APB = \angle BQC = \angle CRD = \angle DSA = 90°$ (property of squares)
$\angle PQR = 90° - y + y = 90° = \angle QRS = \angle RSP = \angle SPQ$
$\therefore$ $PQRS$ is a square