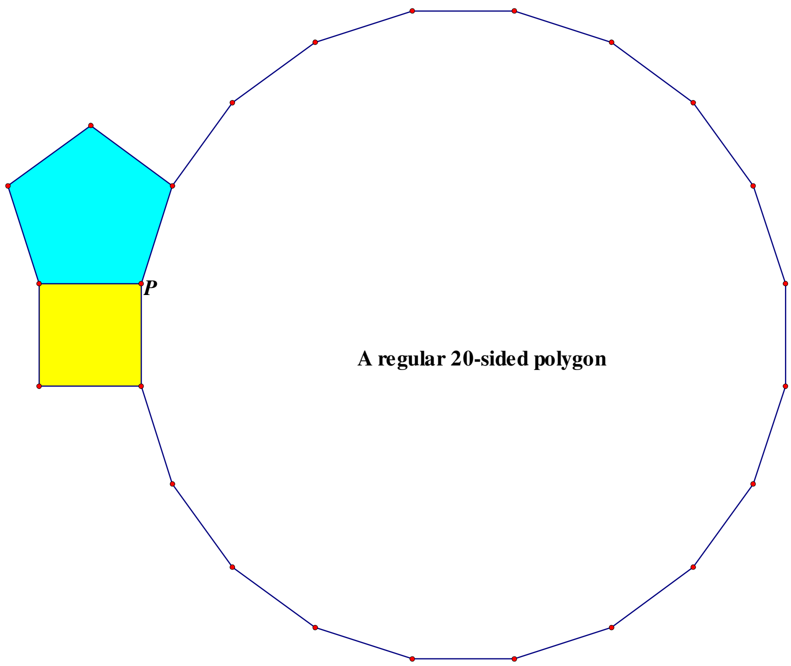
To tessellate a given point *P* by three regular polygons.

Created by Mr. Francis Hung on 20180921. Last updated:01/09/2021



Given a fixed point *P*. The above figure shows a square, a regular pentagon and a regular 20-sided polygon tessellate *P*. Find all possible answers.

Suppose a regular *a*-sided polygon, a regular *b*-sided polygon, and a regular *c*-sided polygon tessellate *P*, where $a \le b \le c$.

Then the three interior angles at *P* are: $180^\circ - \frac{360^\circ}{a}$, $180^\circ - \frac{360^\circ}{b}$ and $180^\circ - \frac{360^\circ}{c}$.

$\therefore\ 180^\circ - \frac{360^\circ}{a} + 180^\circ - \frac{360^\circ}{b} + 180^\circ - \frac{360^\circ}{c} = 360^\circ$ (∠s at a point)

$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}$

If $a = 3$, then $\frac{1}{b} + \frac{1}{c} = \frac{1}{6}$

$6(b + c) = bc$

$bc - 6b - 6c = 0$

$bc - 6b - 6c + 36 = 36$

$(b - 6)(c - 6) = 36$

$b - 6$	$c - 6$	b	c
1	36	7	42
2	18	8	24
3	12	9	18
4	9	10	15
6	6	12	12

If $a = 4$, then $\frac{1}{b} + \frac{1}{c} = \frac{1}{4}$

$4(b + c) = bc$

$bc - 4b - 4c = 0$

$bc - 4b - 4c + 16 = 16$

$(b - 4)(c - 4) = 16$

$b - 4$	$c - 4$	b	c
1	16	5	20
2	8	6	12
4	4	8	8

If $a = 5$, then $\frac{1}{b}+\frac{1}{c}=\frac{3}{10}$

$10(b + c) = 3bc$

$3bc - 10b - 10c = 0$

$9bc - 30b - 30c + 100 = 100$

$3b(3c - 10) - 10(3c - 10) = 100$

$(3b - 10)(3c - 10) = 100$

$3b - 10$	$3c - 10$	b	c
1	100	no integral solution	
2	50	4 ($b < a$, rejected)	20
4	25	no integral solution	
5	20	5	10
10	10	no integral solution	

If $a = 6$, then $\frac{1}{b}+\frac{1}{c}=\frac{1}{3}$ ...... (*)

$bc - 3b - 3c = 0$

$bc - 3b - 3c + 9 = 9$

$(b - 3)(c - 3) = 9$

$b - 3$	$c - 3$	b	c
1	9	4 ($b < a$, rejected)	12
3	3	6	6

If $a \ge 7$, then $\frac{1}{a}<\frac{1}{6}$, $\frac{1}{b}<\frac{1}{6}$, $\frac{1}{c}<\frac{1}{6} \Rightarrow 180^\circ-\frac{360^\circ}{a} > 120^\circ$, $180^\circ-\frac{360^\circ}{b} > 120^\circ$, $180^\circ-\frac{360^\circ}{c} > 120^\circ$

$360^\circ = 180^\circ-\frac{360^\circ}{a}+180^\circ-\frac{360^\circ}{b}+180^\circ-\frac{360^\circ}{c} > 120^\circ + 120^\circ + 120^\circ = 360^\circ$, contradiction

In conclusion:

a	b	c	Interior $\angle$ of a	Interior $\angle$ of b	Interior $\angle$ of c
3	7	42	60°	$\frac{900^\circ}{7}$	$\frac{1200^\circ}{7}$
3	8	24	60°	135°	165°
3	9	18	60°	140°	160°
3	10	15	60°	144°	156°
3	12	12	60°	150°	150°
4	5	20	90°	108°	162°
4	6	12	90°	120°	150°
4	8	8	90°	135°	135°
5	5	10	108°	108°	144°
6	6	6	120°	120°	120°

To tessellate a given point *P* by four regular polygons

Suppose a regular *a*-sided polygon, a regular *b*-sided polygon, a regular *c*-sided polygon and a regular *d*-sided polygon tessellate *P*, where $a \le b \le c \le d$.

The four interior angles at *P* are: $180^\circ - \frac{360^\circ}{a}$, $180^\circ - \frac{360^\circ}{b}$, $180^\circ - \frac{360^\circ}{c}$ and $180^\circ - \frac{360^\circ}{d}$.

$\therefore\ 180^\circ - \frac{360^\circ}{a} + 180^\circ - \frac{360^\circ}{b} + 180^\circ - \frac{360^\circ}{c} + 180^\circ - \frac{360^\circ}{d} = 360^\circ$ (∠s at a point)

$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 1$

If $(a, b) = (3, 3)$, then $\frac{1}{c} + \frac{1}{d} = \frac{1}{3}$

By the result of (*), $(c, d) = (6, 6)$.

If $(a, b) = (3, 4)$, then $\frac{1}{c} + \frac{1}{d} = \frac{5}{12}$

$12(c + d) = 5cd$

$5cd - 12c - 12d = 0$

$25cd - 60c - 60d + 144 = 144$

$(5c - 12)(5d - 12) = 144$

$5c - 12$	$5d - 12$	c	d
1	144	no integral solution	
2	72	no integral solution	
3	48	3 ($c < b$, rejected)	
4	36	no integral solution	
6	24	no integral solution	
8	18	4	6
9	16	no integral solution	
12	12	no integral solution	

If $a = 3$, $b \ge 5$, then $c \ge 5$, $d \ge 5$, the interior angles are 60°, *B*, *C*, *D*, where $B, C, D \ge 108^\circ$.

Sum of angle at $P \ge (60 + 3\times108)^\circ = 384^\circ$. This violates the fact ∠s at a point = 360°.

If $(a, b) = (4, 4)$, then $\frac{1}{c} + \frac{1}{d} = \frac{1}{2}$ …… (**)

$2(c + d) = cd$

$cd - 2c - 2d + 4 = 4$

$(c - 2)(d - 2) = 4$

$c - 2$	$d - 2$	c	d
1	4	3 ($c < b$, rejected)	6
2	2	4	4

In conclusion:

a	*b*	*c*	*d*	Interior ∠ of *a*	Interior ∠ of *b*	Interior ∠ of *c*	Interior ∠ of *d*
3	3	6	6	60°	60°	120°	120°
3	4	4	6	60°	90°	90°	120°
4	4	4	4	90°	90°	90°	90°

To tessellate a given point *P* by five regular polygons

Suppose a regular *a*-sided polygon, a regular *b*-sided polygon, a regular *c*-sided polygon, a regular *d*-sided polygon and a regular *e*-sided polygon tessellate *P*, where $a \le b \le c \le d \le e$.

The interior angles at *P* are: $180^\circ - \frac{360^\circ}{a}, 180^\circ - \frac{360^\circ}{b}, 180^\circ - \frac{360^\circ}{c}, 180^\circ - \frac{360^\circ}{d}$ and $180^\circ - \frac{360^\circ}{e}$.

$\therefore\ 180^\circ - \frac{360^\circ}{a} + 180^\circ - \frac{360^\circ}{b} + 180^\circ - \frac{360^\circ}{c} + 180^\circ - \frac{360^\circ}{d} + 180^\circ - \frac{360^\circ}{e} = 360^\circ$ (∠s at a point)

$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} = \frac{3}{2}$

If $(a, b, c) = (3, 3, 3)$, then $\frac{1}{d} + \frac{1}{e} = \frac{1}{2}$

By the result of (**), $(d, e) = (3, 6)$ or $(4, 4)$.

If $(a,b) = (3,3)$ and $c \ge 4$, then $d \ge 4$, $e \ge 4$, the interior ∠s are 60°, 60°, *C*, *D*, *E*, where *C*, *D*, *E* > 90°

Sum of angle at $P \ge 60^\circ + 60^\circ + 90^\circ \times 3 = 390^\circ$. This violates the fact ∠s at a point = 360°.

In conclusion:

a	*b*	*c*	*d*	*e*	Interior ∠ of *a*	Interior ∠ of *b*	Interior ∠ of *c*	Interior ∠ of *d*	Interior ∠ of *e*
3	3	3	3	6	60°	60°	60°	60°	120°
3	3	3	4	4	60°	60°	60°	90°	90°

To tessellate a given point *P* by six regular polygons

Suppose the regular polygons with the number of sides: *a*, *b*, *c*, *d*, *e* and *f* tessellate *P*, where $a \le b \le c \le d \le e \le f$.

If $a = b = c = d = e = f = 3$, then the interior angles are all equal to 60°.

The angle sum at $P = 6 \times 60^\circ = 360^\circ$. This is a possible solution.

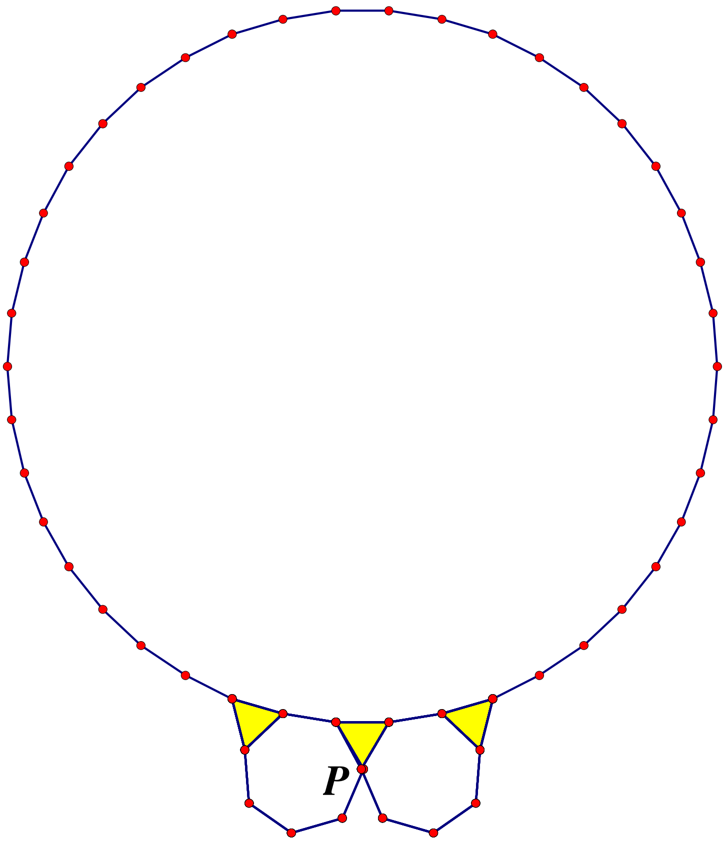
If $f > 3$, then the sum of angles at $P > 360^\circ$, which is impossible.

There are no other solutions.

It is impossible to tessellate a given point *P* by more than six regular polygons.

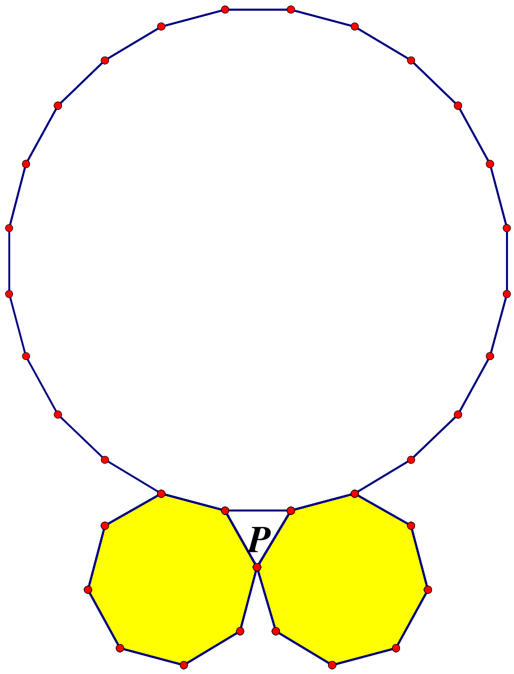
To tessellate a plane by regular polygons with the same vertex configuration at every point

a	*b*	*c*	Interior ∠ of *a*	Interior ∠ of *b*	Interior ∠ of *c*
3	7	42	60°	$\frac{900^\circ}{7}$	$\frac{1200^\circ}{7}$



Sum of interior angles at point $P = 60° + \frac{900^\circ}{7} + \frac{900^\circ}{7} = \frac{2220^\circ}{7}$ Impossible

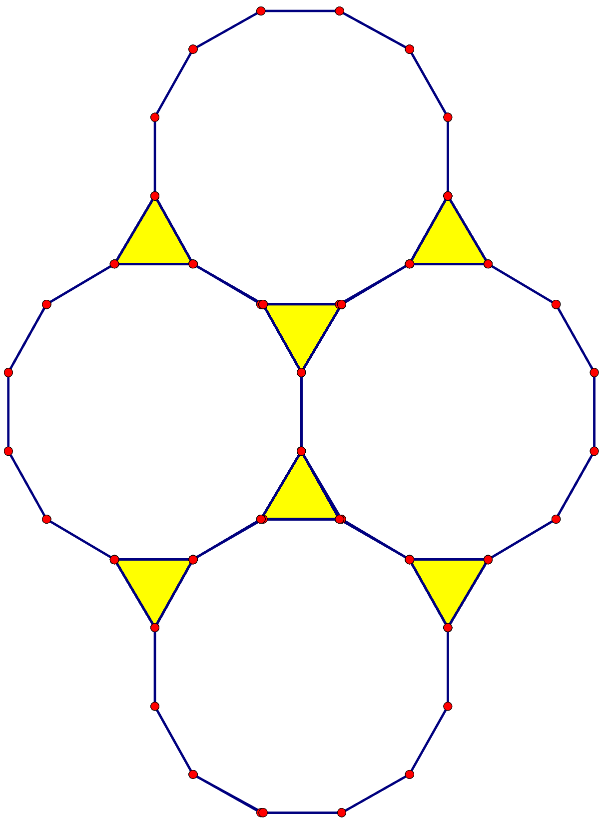
a	*b*	*c*	Interior ∠ of *a*	Interior ∠ of *b*	Interior ∠ of *c*
3	8	24	60°	135°	165°



Sum of interior angles at point P = 60° +135°+135° = 330° Impossible

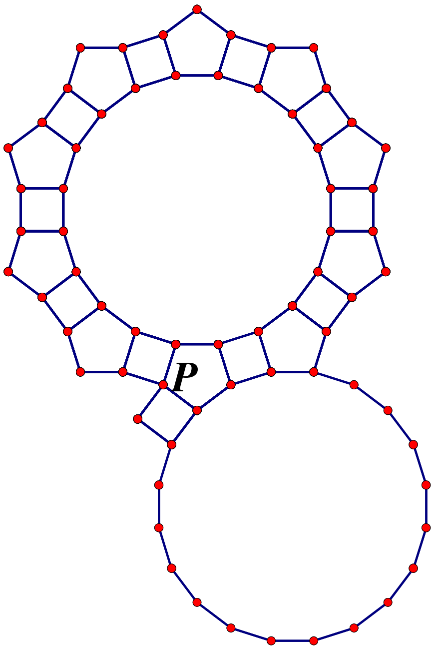
Hence, the bottom angle at *P* must be tessellated by the 3 interior angles of *a*, *b* and *c*.

a	*b*	*c*	Interior ∠ of *a*	Interior ∠ of *b*	Interior ∠ of *c*	sum of angles at *P*	conclusion
3	9	18	60°	140°	160°	60°+140°+140°=340°	impossible
3	10	15	60°	144°	156°	60°+144°+144°=348°	impossible
3	12	12	60°	150°	150°	60°+150°+150°=360°	possible

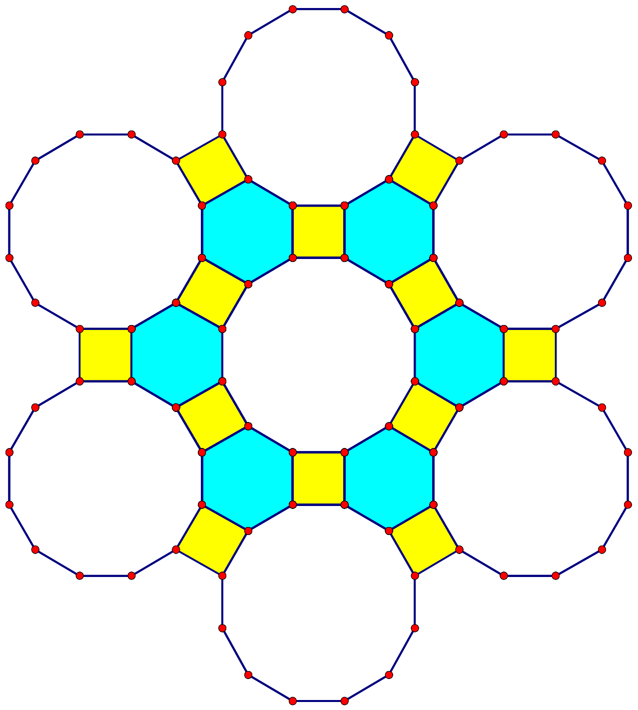


vertex configuration [3,12,12]

a	*b*	*c*	Interior ∠ of *a*	Interior ∠ of *b*	Interior ∠ of *c*	sum of angles at *P*	conclusion
4	5	20	90°	108°	162°	90°+108°+90°=288°	impossible

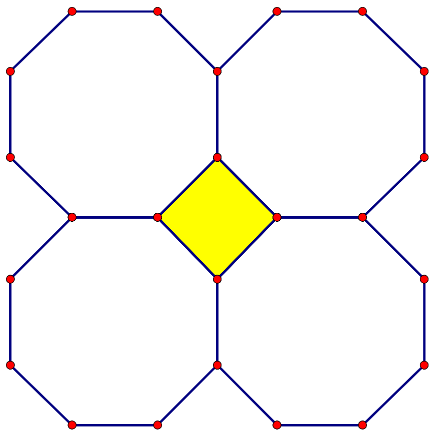


a	b	c	Interior ∠ of a	Interior ∠ of b	Interior ∠ of c
4	6	12	90°	120°	150°



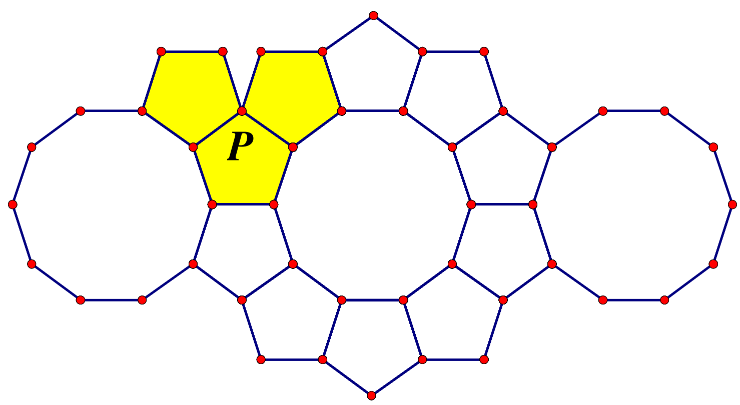
vertex configuration [4,6,12]

a	b	c	Interior ∠ of a	Interior ∠ of b	Interior ∠ of c	sum of angles at P	conclusion
4	8	8	90°	135°	135°	90°+135°+135°=360°	possible

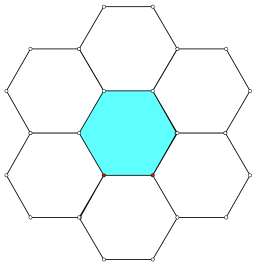


vertex configuration [4,8,8]

a	b	c	Interior ∠ of a	Interior ∠ of b	Interior ∠ of c	sum of angles at P	conclusion
5	5	10	108°	108°	144°	108°+108°+108°=324°	impossible

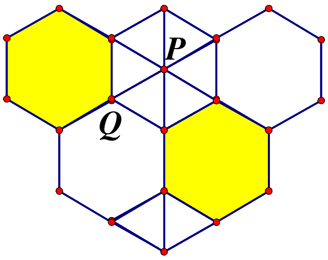


a	b	c	Interior ∠ of a	Interior ∠ of b	Interior ∠ of c	sum of angles at P	conclusion
6	6	6	120°	120°	120°	120°+120°+120°=360°	possible

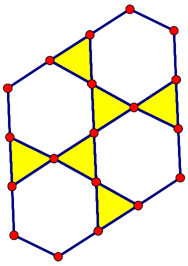


vertex configuration [6,6,6]

a	b	c	d	Interior ∠ of a	Interior ∠ of b	Interior ∠ of c	Interior ∠ of d
3	3	6	6	60°	60°	120°	120°

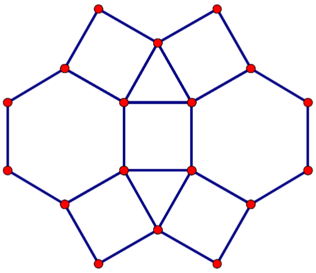


Although the regular polygons can tessellate the plane, the vertex configurations at P and Q are [3,3,3,3,3,3] and [3,3,6,6]. So it is not a semi-regular tessellation.

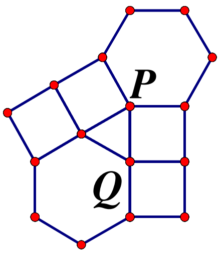


vertex configuration [3,6,3,6]

a	b	c	d	Interior ∠ of a	Interior ∠ of b	Interior ∠ of c	Interior ∠ of d
3	4	4	6	60°	90°	90°	120°

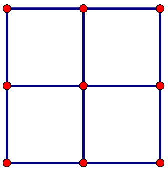


vertex configuration [3,4,6,4]



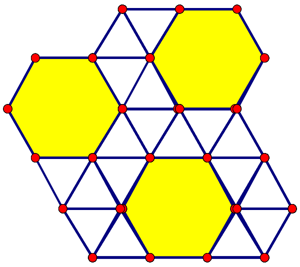
Although the regular polygons can tessellate the plane, the vertex configurations at P and Q are [3,4,6,4] and [3,4,4,6]. So it is not a semi-regular tessellation.

a	b	c	d	Interior ∠ of a	Interior ∠ of b	Interior ∠ of c	Interior ∠ of d
4	4	4	4	90°	90°	90°	90°



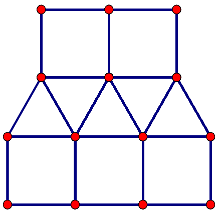
vertex configuration [4,4,4,4]

a	b	c	d	e	Interior ∠ of a	Interior ∠ of b	Interior ∠ of c	Interior ∠ of d	Interior ∠ of e
3	3	3	3	6	60°	60°	60°	60°	120°

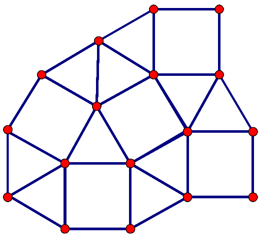


vertex configuration [3,3,3,3,6]

a	b	c	d	e	Interior ∠ of a	Interior ∠ of b	Interior ∠ of c	Interior ∠ of d	Interior ∠ of e
3	3	3	4	4	60°	60°	60°	90°	90°

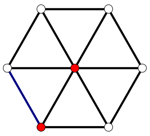


vertex configuration [3,3,3,4,4] or



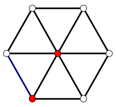
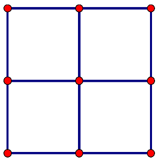
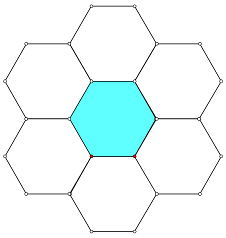
[3,3,4,3,4]

If $a = b = c = d = e = f = 3$, then the interior angles are all equal to 60°.

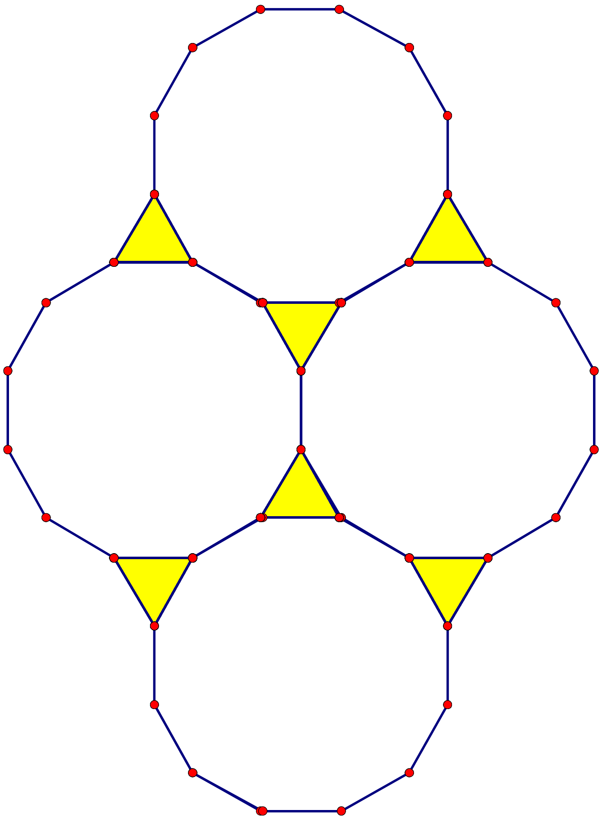
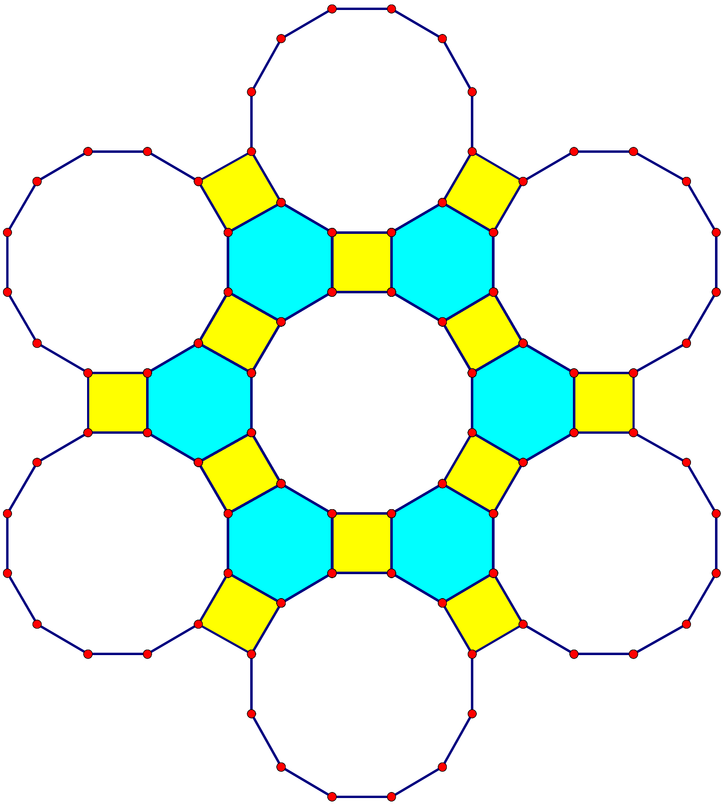


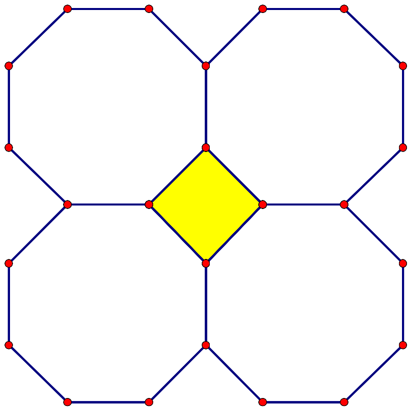
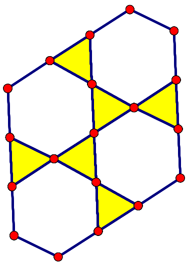
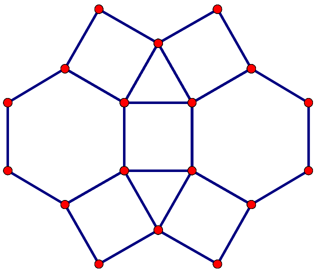
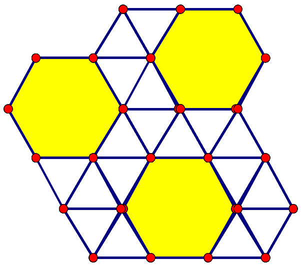
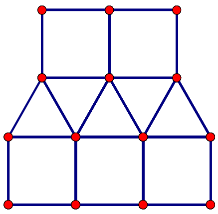
vertex configuration [3,3,3,3,3,3]

Conclusion Regular tessellation by a single polygon

Types of polygon	vertex configuration	Diagram
Equilateral triangles	[3,3,3,3,3,3]	
Squares	[4,4,4,4]	
Regular hexagon	[6,6,6]	

Semi-regular tessellation by more than one polygon (same vertex configuration at every point)

vertex configuration	Diagram
[3,12,12]	
[4,6,12]	

[4,8,8]	
[3,6,3,6]	
[3,4,6,4]	
[3,3,3,3,6]	
[3,3,3,4,4]	
[3,3,4,3,4]