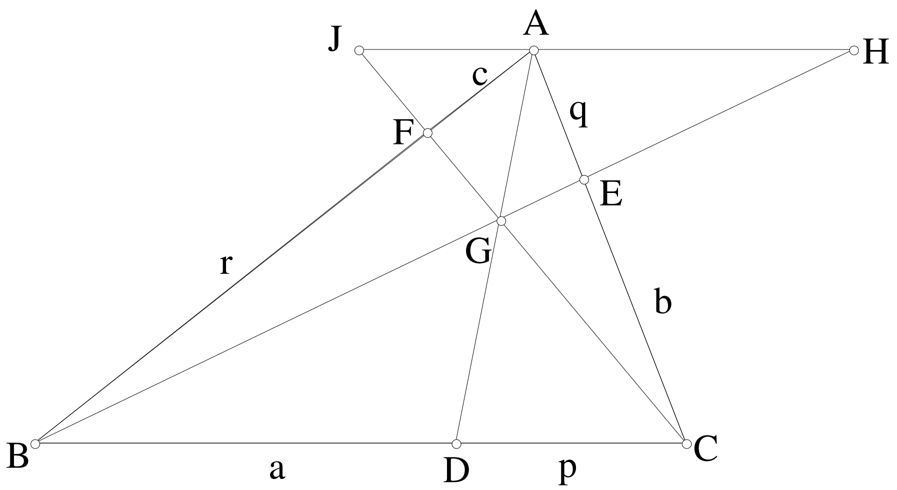
Ceva's Theorem

Reference: Advanced Level Pure Mathematics by S.L.Green p. 130-131.

Created by Mr. Francis Hung on 1 July 2008 Last updated: 2021-09-02

In ΔABC, D, E, F are points on BC, CA and AB respectively. AD, BE and CF are concurrent at G. Suppose that $BD = a$, $DC = p$, $CE = b$, $EA = q$, $AF = c$, $FB = r$. Then $\frac{a}{p}\cdot\frac{b}{q}\cdot\frac{c}{r}=1$

Proof:



Construct a line JAH parallel to BDC. Produce CF and BH to meet the parallel line.

$\Delta AFJ \sim \Delta BFC \Rightarrow \frac{c}{r}=\frac{AJ}{a+p}$ ...... (1)

$\Delta AEH \sim \Delta CEB \Rightarrow \frac{b}{q}=\frac{a+p}{AH}$ ...... (2)

$\Delta AJG \sim \Delta CDG \Rightarrow \frac{AJ}{p}=\frac{AG}{GD}$ ...... (3)

$\Delta AHG \sim \Delta DBG \Rightarrow \frac{a}{AH}=\frac{GD}{AG}$ ...... (4)

$(1)\times(2)\times(3)\times(4)$ $\frac{c}{r}\cdot\frac{b}{q}\cdot\frac{AJ}{p}\cdot\frac{a}{AH}=\frac{AJ}{a+p}\cdot\frac{a+p}{AH}\cdot\frac{AG}{GD}\cdot\frac{GD}{AG}$

$\Rightarrow\frac{a}{p}\cdot\frac{b}{q}\cdot\frac{c}{r}=1$

The theorem is proved.

Converse of Ceva's Theorem

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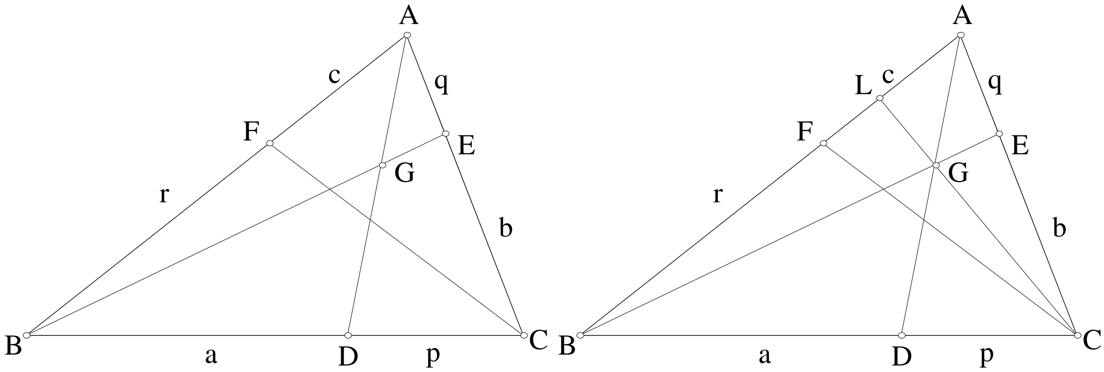
Suppose $\frac{a}{p}\cdot\frac{b}{q}\cdot\frac{c}{r}=1$, then *AD*, *BE* and *CF* are concurrent at a point *G*.



The proof is easy.

Suppose the three lines *AD*, *BE* and *CF* are not concurrent.

Let *AD* and *BE* intersect at *G*. Produce *CG* to meet *AB* at *L*.

Then by Ceva's Theorem, $\frac{a}{p}\cdot\frac{b}{q}\cdot\frac{AL}{LB}=1$.

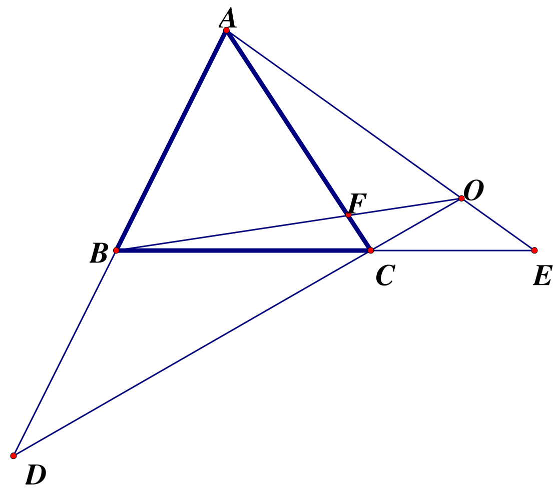
Given that $\frac{a}{p}\cdot\frac{b}{q}\cdot\frac{c}{r}=1$.

Compare these two equations, we have $\frac{AL}{LB}=\frac{c}{r}$

which means that *L* = *F* and the three lines are concurrent.

Another proof of Ceva's Theorem and its converse

(a) Let h_1 be the height of ΔABC with AB as its base.



$$\frac{\text{Area of }\Delta ADC}{\text{Area of }\Delta BDC}=\frac{\frac{1}{2}AD\cdot h_1}{\frac{1}{2}DB\cdot h_1}=\frac{AD}{DB}$$

Let h_2 be the height of ΔABO with AB as its base.

$$\frac{\text{Area of }\Delta ADO}{\text{Area of }\Delta BDO}=\frac{\frac{1}{2}AD\cdot h_2}{\frac{1}{2}DB\cdot h_2}=\frac{AD}{DB}$$

$$\frac{AD}{DB}=\frac{\text{Area of }\Delta ADC}{\text{Area of }\Delta BDC}=\frac{\text{Area of }\Delta ADO}{\text{Area of }\Delta BDO}$$

$$=\frac{\text{Area of }\Delta ADO-\text{Area of }\Delta ADC}{\text{Area of }\Delta BDO-\text{Area of }\Delta BDC}$$

$$=\frac{\text{Area of }\Delta OAC}{\text{Area of }\Delta OBC}$$

Similarly, $\frac{BE}{EC}=\frac{\text{Area of }\Delta ABE}{\text{Area of }\Delta ACE}=\frac{\text{Area of }\Delta OBE}{\text{Area of }\Delta OCE}$

$$=\frac{\text{Area of }\Delta ABE-\text{Area of }\Delta OBE}{\text{Area of }\Delta ACE-\text{Area of }\Delta OCE}$$

$$=\frac{\text{Area of }\Delta OAB}{\text{Area of }\Delta OAC}$$

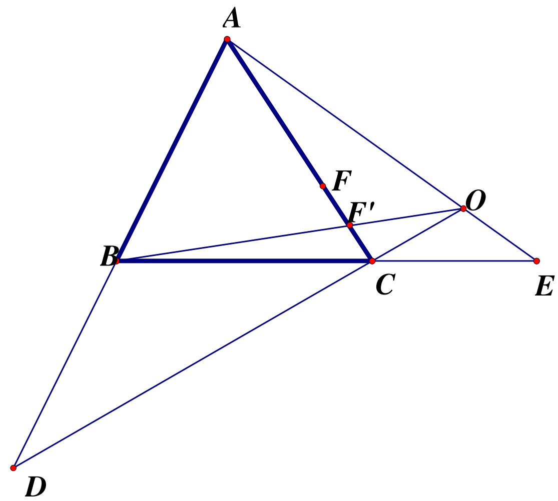
Similarly, $\frac{CF}{FA}=\frac{\text{Area of }\Delta BFC}{\text{Area of }\Delta BFA}=\frac{\text{Area of }\Delta CFO}{\text{Area of }\Delta AFO}$

$$=\frac{\text{Area of }\Delta BFC+\text{Area of }\Delta CFO}{\text{Area of }\Delta BFA+\text{Area of }\Delta AFO}$$

$$=\frac{\text{Area of }\Delta OBC}{\text{Area of }\Delta OAB}$$

$$\frac{AD}{DB}\cdot\frac{BE}{EC}\cdot\frac{CF}{FA}=\frac{\text{Area of }\Delta OAC}{\text{Area of }\Delta OBC}\cdot\frac{\text{Area of }\Delta OAB}{\text{Area of }\Delta OAC}\cdot\frac{\text{Area of }\Delta OBC}{\text{Area of }\Delta OAB}=1$$

(b) If $\frac{AD}{DB}\cdot\frac{BE}{EC}\cdot\frac{CF}{FA}=1$, choose a point F' such that AE, BF' produced and DC produced intersect at a point O, as shown.



According to the result in (a),

$$\frac{AD}{DB}\cdot\frac{BE}{EC}\cdot\frac{CF'}{F'A}=1$$

$$\therefore\ \frac{AD}{DB}\cdot\frac{BE}{EC}\cdot\frac{CF}{FA}=\frac{AD}{DB}\cdot\frac{BE}{EC}\cdot\frac{CF'}{F'A}$$

$$\frac{CF}{FA}=\frac{CF'}{F'A}$$

$$\frac{CF}{FA}+1=\frac{CF'}{F'A}+1$$

$$\frac{CF+FA}{FA}=\frac{CF'+F'A}{F'A}$$

$$\frac{AC}{FA}=\frac{AC}{F'A}$$

$\therefore$ $FA = F'A$

$\therefore$ F and F' are the same point

$\therefore$ AE, BF produced and DC produced intersect at a point O .

Ceva's Theorem Example

Created by Mr. Francis Hung on 1 July 2008

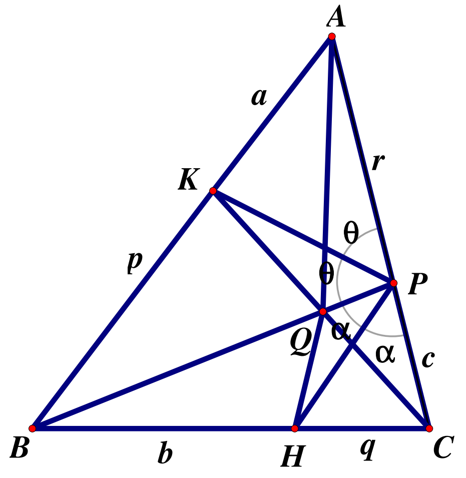
1972 中文中學會考高級數學試卷二 Q7

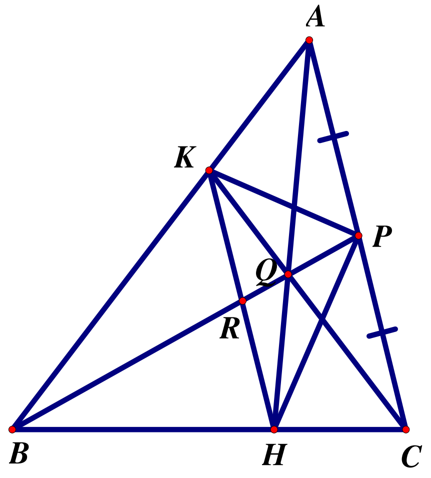
P 為任意三角形 ABC 邊 AC 上一點，若$\angle APB$ 之分角線交 AB 於 K，$\angle BPC$ 之分角線交 BC 於 H。

(i) 證明 AH、BP、CK 三線共點。

(ii) 若 P 為 AC 之中點，證明 BP 平分 HK。

(i)





Suppose CK intersects BP at Q.

Let $AK = a$, $KB = p$, $BH = b$, $HC = q$, $CP = c$, $PA = r$

In ΔAKP, $\dfrac{a}{\sin\theta} = \dfrac{r}{\sin\angle AKP}$ …… (1)

In ΔBKP, $\dfrac{p}{\sin\theta} = \dfrac{BP}{\sin\angle BKP}$ …… (2)

In ΔBHP, $\dfrac{b}{\sin\alpha} = \dfrac{BP}{\sin\angle BHP}$ …… (3)

In ΔCHP, $\dfrac{q}{\sin\alpha} = \dfrac{c}{\sin\angle CHP}$ …… (4)

Note that $\sin\angle AKP = \sin\angle BKP$, $\sin\angle BHP = \sin\angle CHP$ and $\alpha + \theta = 90°$ (adj. $\angle$s on st. line)

$\dfrac{(1)}{(2)}$: $\dfrac{a}{p} = \dfrac{r}{BP}$; $\dfrac{(3)}{(4)}$: $\dfrac{b}{q} = \dfrac{BP}{c}$

Multiply these two equations together: $\dfrac{a}{p}\cdot\dfrac{b}{q} = \dfrac{r}{c}$

$\therefore\ \dfrac{a}{p}\cdot\dfrac{b}{q}\cdot\dfrac{c}{r} = 1$

By the converse of Ceva's Theorem, AH, BP, CK are concurrent at a point.

(ii) If P is the mid point of AC, then $c = r$. Suppose KH intersects BP at R.

By the above result, $\dfrac{a}{p}\cdot\dfrac{b}{q} = 1 \Rightarrow \dfrac{a}{p} = \dfrac{q}{b} \Rightarrow \dfrac{a}{p} + 1 = \dfrac{q}{b} + 1 \Rightarrow \dfrac{a+p}{p} = \dfrac{q+b}{b} \Rightarrow \dfrac{BA}{BK} = \dfrac{BC}{BH}$

$\therefore\ \Delta BHK \sim \Delta BCA$

$\Rightarrow \angle BHK = \angle BCA$ (corr. $\angle$s ~ Δ's)

$\Rightarrow KH \,//\, AC$ (corr. $\angle$s eq.)

$\Rightarrow \Delta BKR \sim \Delta BAP$ and $\Delta BHR \sim \Delta BCP$ (equiangular)

$\dfrac{KR}{AP} = \dfrac{BR}{BP} = \dfrac{RH}{PC}$ (ratio of sides, ~Δ's)

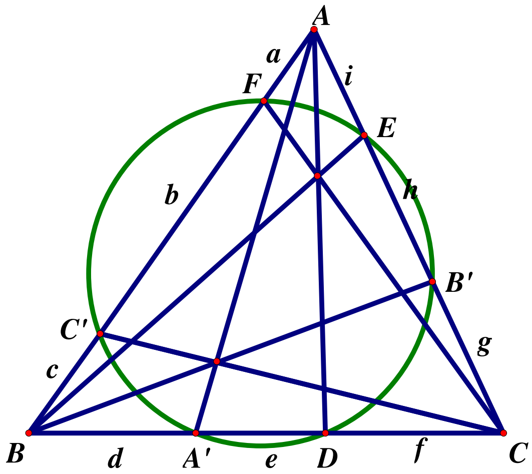
$\because AP = PC$

$\therefore\ KR = RH$, i.e. BP bisects HK at R.

1973 中文中學會考高級數學試卷二 Q8(ii) 1968 香港中文中學會考高級數學試卷二 Q3(iii)

(ii) 由三角形 ABC 之各頂點至對邊引三共點線 AA', BB' 及 CC'；過 A'、B' 及 C' 作一圓與三角形之三邊 BC、CA、AB 依次交於 D、E、F。求證 AD、BE、CF 三線共點。

(ii) Let $AF = a$, $FC' = b$, $C'B = c$, $BA' = d$, $A'D = e$, $DC = f$, $CB' = g$, $B'E = h$, $EA = i$



By Ceva's Theorem, $\frac{a+b}{c}\cdot\frac{d}{e+f}\cdot\frac{g}{h+i}=1\ldots$ (1)

By intersecting chords theorem,

$a(a + b) = i(h + i)$ … (2)

$c(b + c) = d(d + e)$ … (3)

$f(e + f) = g(g + h)$ … (4)

From (2): $\frac{a+b}{h+i}=\frac{i}{a}$

From (3): $\frac{d}{c}=\frac{b+c}{d+e}$

From (4): $\frac{g}{e+f}=\frac{f}{g+h}$

Multiply these 3 equations: $\frac{a+b}{c}\cdot\frac{d}{e+f}\cdot\frac{g}{h+i}=\frac{i}{a}\cdot\frac{b+c}{d+e}\cdot\frac{f}{g+h}$

By (1): $1=\frac{i}{a}\cdot\frac{b+c}{d+e}\cdot\frac{f}{g+h}$

$\therefore\ \frac{a}{b+c}\cdot\frac{d+e}{f}\cdot\frac{g+h}{i}=1$

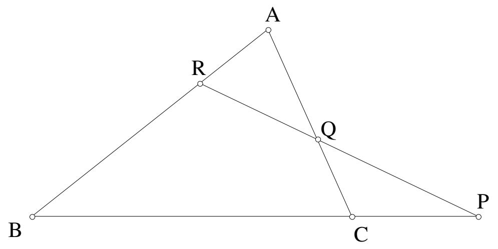
By the converse of Ceva's Theorem, AD, BE and CF are concurrent at a point.

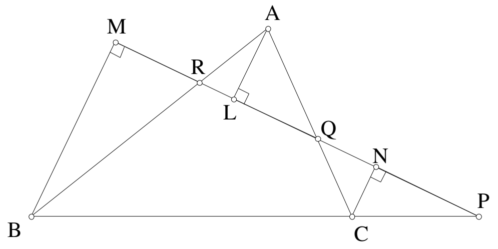
Menelaus's Theorem

Reference: Advanced Level Pure Mathematics by S.L.Green p. 132-133.

Created by Francis Hung on 3 July 2008 Last updated: : 02 September 2021

In ΔABC, suppose a line cuts BC at P, AC at Q and AB at R, then $\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB}=-1$





Let L, M, N are points on the line PQR produced such that AL, BM, $CN \perp PQR$.

$\Delta BMR \sim \Delta ALR \Rightarrow \frac{AR}{BR}=\frac{AL}{BM}$ ...... (1)

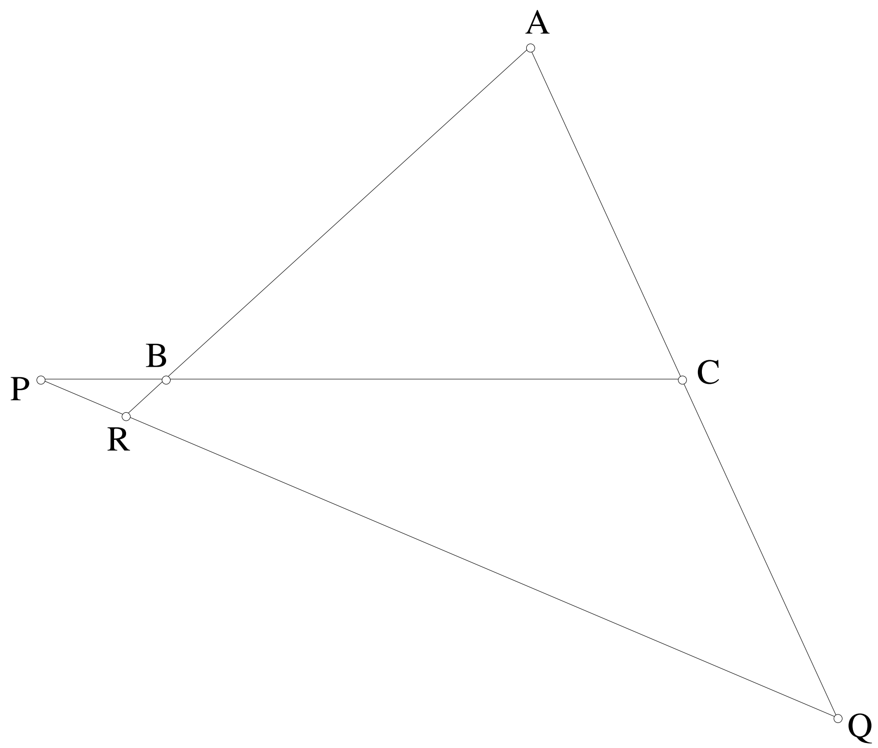
$\Delta CNQ \sim \Delta ALQ \Rightarrow \frac{CQ}{AQ}=\frac{CN}{AL}$ ...... (2)

$\Delta BMP \sim \Delta CNP \Rightarrow \frac{BP}{CP}=\frac{BM}{CN}$ ...... (3)

$(1)\times(2)\times(3) \Rightarrow \frac{BP}{CP}\cdot\frac{CQ}{AQ}\cdot\frac{AR}{BR}=1$

Remark: If we consider the direction, $\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB}=-1$.

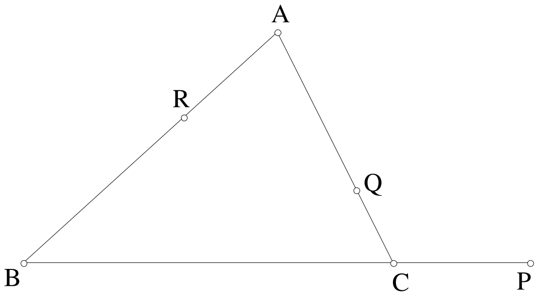
Exercise: Investigate the follow case and verify **Menelaus's Theorem:**

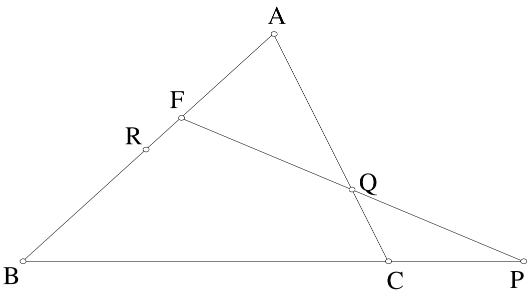


Converse of Menelaus's Theorem

Created by Francis Hung on 3 July 2008 Last updated: 02 September 2021

If points P, Q, R on the sides BC, CA, AB respectively are such that $\frac{BP}{CP}\cdot\frac{CQ}{AQ}\cdot\frac{AR}{BR}=1$, then P, Q, R are collinear.





Join PQ, and produce it to meet AB at F.

By Menelaus' Theorem, $\frac{BP}{CP}\cdot\frac{CQ}{AQ}\cdot\frac{AF}{BF}=1$

Compare with $\frac{BP}{CP}\cdot\frac{CQ}{AQ}\cdot\frac{AR}{BR}=1$ (given)

we have $\frac{AR}{BR}=\frac{AF}{BF}$.

So $R = F$ and R coincides with F and that the 3 points P, Q, R are collinear.

Menelaus's Theorem Example

1973 中文中學會考高級數學試卷二 Q8(i)

(i) 設三角形 ABC 三邊 BC、CA 及 AB 之中點依次為 D、E 及 F。AD 與 EF 交於 M，CM 與 AB 交於 N。求證：$AB = 3AN$。

(i) Let P be the mid point of CD.

Join EP and FP. Suppose NC intersects EP at J, FP intersects AD at K.

$FE // BC$ and $FE = \frac{1}{2}BC$ (mid point theorem)

It is easy to show that $\Delta AFM \sim \Delta ABD$ and $\Delta AEM \sim \Delta ACD$ (equiangular)

$\frac{FM}{BD} = \frac{AF}{AB} = \frac{1}{2}$ (ratio of sides, ~ Δ's) $\Rightarrow FM = \frac{1}{2}BD$

$\frac{ME}{DC} = \frac{AE}{AC} = \frac{1}{2}$ (ratio of sides, ~ Δ's) $\Rightarrow ME = \frac{1}{2}CD$

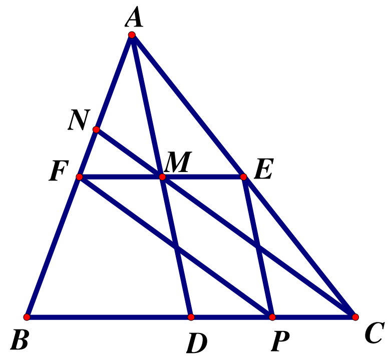
$\because BD = DC \therefore FM = ME$

$\because P$ is the mid point of CD

$\therefore CP = PD = FM = ME$

$\therefore MEPD$ and $CMFP$ are //-grams (opp. sides are eq. and parallel)

$\therefore AD // EP$ and $NC // FP$ (property of //-grams)



$NF : FB = CP : PB$

$NF : \frac{1}{2}AB = \frac{1}{2}CD : (BD + DP)$

$2NF : AB = CD : 2(BD + \frac{1}{2}CD)$

$2NF : AB = \frac{1}{2}BC : 2\times\frac{3}{4}BC$

$2NF : AB = 1 : 3$

$NF = \frac{1}{6}AB$

$AN = AF - NF = \frac{1}{2}AB - \frac{1}{6}AB = \frac{1}{3}AB$

$\therefore AB = 3AN$

Method 2

Regard the line NMC as an intercept of ΔABD.

Apply Menelaus' Theorem, $\frac{AN}{NB}\cdot\frac{BC}{CD}\cdot\frac{DM}{MA} = -1$

$\frac{AN}{NB}\cdot\frac{2}{-1}\cdot\frac{1}{1} = -1$

$\therefore NB = 2AN$

$AB = 3AN$

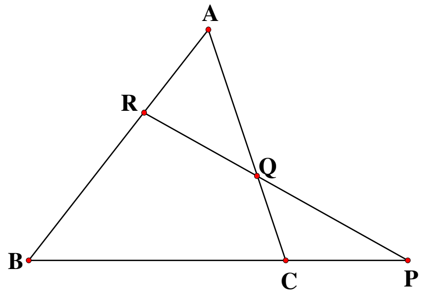
1970 香港中文中學會考高級數學試卷二 Q8

(i) 試述孟氏(Menelaus)定理及其逆定理。(不需證明)

(ii) 若 $ABCD$ 為一長方形。L、M、N、P 依次在 AB、BC、CD、DA 上，且 $PM // AB$ 及 $LN // BC$。若 LM 、PN 之延線交於 K，試證 K 必在 AC 之延線上。

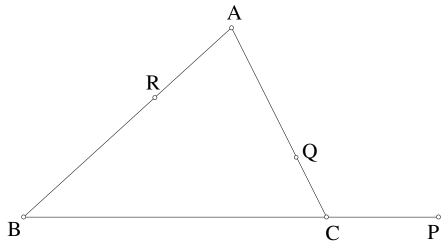
(i) Menelaus's theorem

In ΔABC, suppose a line cuts BC at P, AC at Q and AB at R, then $\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB}=-1$



Converse of Menelaus's theorem

If points P, Q, R on the sides BC, CA, AB respectively are such that $\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB}=-1$, then P, Q, R are collinear.

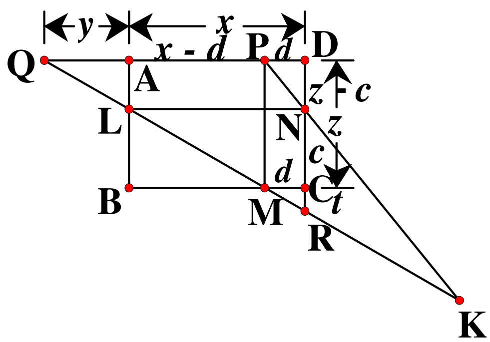


(ii) Extend DA to cut RL produced at Q.

Extend DC to cut QK at R.

$AD = x$, $AQ = y$, $DC = z$, $CR = t$, $NC = c$, $DP = d$

Consider the intercept RNK on ΔDQR.



$$\frac{DP}{PQ}\cdot\frac{QK}{KR}\cdot\frac{RN}{ND}=-1 \quad \text{(Menelaus's theorem)}$$

$$\frac{d}{x-d+y}\cdot\frac{QK}{KR}\cdot\frac{c+t}{z-c}=-1$$

$$\frac{QK}{KR}=-\frac{z-c}{c+t}\cdot\frac{x-d+y}{d} \cdots\cdots (1)$$

It is easy to show that $\Delta QAL \sim \Delta QPM \sim \Delta MCR \sim \Delta LNR$

$$\frac{z-c}{y}=\frac{z}{x+y-d}=\frac{t}{d}=\frac{c+t}{x}=k \text{ (corr. sides, \sim}\Delta\text{s)} \cdots\cdots (2)$$

Consider the points A, C, K on ΔDQR.

$$\frac{DA}{AQ}\cdot\frac{QK}{KR}\cdot\frac{RC}{CD}=-\frac{x}{y}\cdot\frac{z-c}{c+t}\cdot\frac{x-d+y}{d}\cdot\frac{t}{z} \quad \text{by (1)}$$

$$=-\frac{x}{c+t}\cdot\frac{z-c}{y}\cdot\frac{x+y-d}{z}\cdot\frac{t}{d}$$

$$=-\frac{1}{k}\cdot k\cdot\frac{1}{k}\cdot k=-1 \quad \text{by (2)}$$

$\therefore$ By the converse if Menelaus's theorem, A, C, K are collinear.

Theorem on a triangle

By Mr. Francis Hung Last updated: 21 April 2011

In ΔABC, AD, BF and CE are concurrent at G.

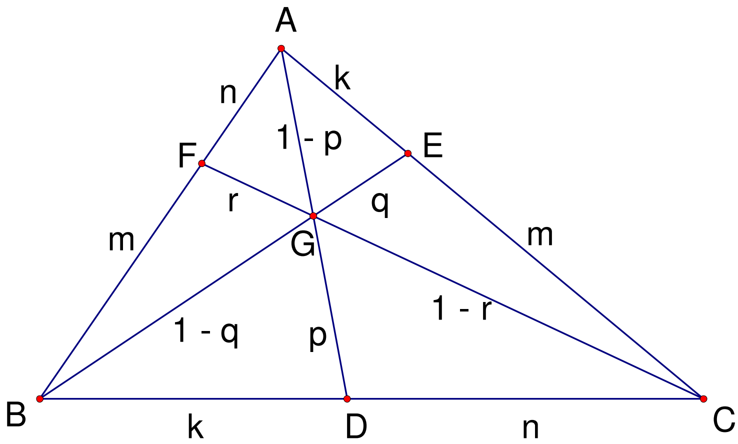
By Ceva's theorem, $\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1$

Let $BD : DC = k : n$, $CF : FP = m : k$, $AE : EB = n : m$

Let $AG : GD = 1 - p : p$, $BG : GE = 1 - q : q$, $CG : GF = 1 - r : r$

Then $p + q + r = 1$

Proof: method 1 (vector method)



Let $\overrightarrow{BC}=\vec{c}$, $\overrightarrow{BA}=\vec{a}$, $\overrightarrow{BD}=\frac{k}{k+n}\vec{c}$

$$\overrightarrow{BG}=p\overrightarrow{BA}+(1-p)\overrightarrow{BD}=p\,\vec{a}+\frac{(1-p)k\vec{c}}{k+n}$$

$$\overrightarrow{BE}=\frac{m\vec{a}+k\vec{c}}{m+k}$$

$\because \overrightarrow{BG}\,//\,\overrightarrow{BE}$, $\overrightarrow{BG}=s\,\overrightarrow{BE}$

$$p\,\vec{a}+\frac{(1-p)k\vec{c}}{k+n}=\frac{sm\vec{a}}{m+k}+\frac{sk\vec{c}}{m+k}$$

Compare coefficients,

$$\therefore\begin{cases}\frac{sm}{m+k}=p\cdots\cdots\cdots\cdots(1)\\ \frac{sk}{m+k}=\frac{(1-p)k}{k+n}\cdots(2)\end{cases}$$

$$\frac{(2)}{(1)}:\ \frac{k}{m}=\frac{(1-p)k}{p(k+n)}$$

$pk + pn = m - mp$

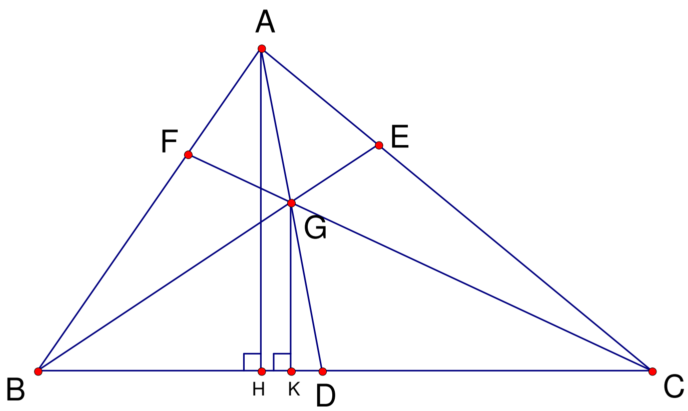
$p(k + m + n) = m$

$$p=\frac{m}{k+m+n}$$

similarly $q=\frac{n}{k+m+n}$, $r=\frac{k}{k+m+n}$

$\therefore p + q + r = 1$

Method 2



Draw AH, $GK \perp BC$

$$\frac{S_{\Delta BGC}}{S_{\Delta ABC}}=\frac{\frac{1}{2}BC\cdot GK}{\frac{1}{2}BC\cdot AH}=\frac{GK}{AH}=\frac{GD}{AD}\quad\cdots\cdots(1)$$

similarly $\frac{S_{\Delta AGC}}{S_{\Delta ABC}}=\frac{GE}{BE}\quad\cdots\cdots\cdots(2)$

and $\frac{S_{\Delta AGB}}{S_{\Delta ABC}}=\frac{GF}{CF}\quad\cdots\cdots\cdots(3)$

(1) + (2) + (3)

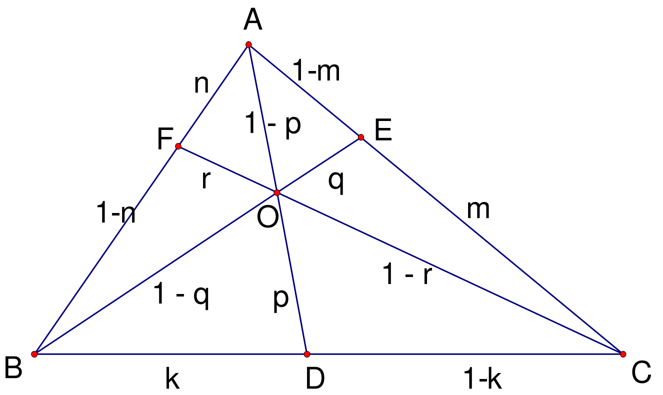
$$\frac{S_{\Delta BGC}+S_{\Delta AGC}+S_{\Delta AGB}}{S_{\Delta ABC}}=\frac{GD}{AD}+\frac{GE}{BE}+\frac{GF}{CF}$$

$$\frac{GD}{AD}+\frac{GE}{BE}+\frac{GF}{CF}=\frac{S_{\Delta ABC}}{S_{\Delta ABC}}=1$$

$\therefore p + q + r = 1$

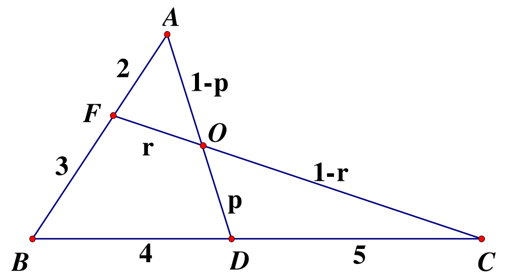
Discussion:

In ΔABC, AD, BF and CE are concurrent at O.
$AO : OD = 1 - p : p$; $BO : OE = 1 - q : q$;
$CO : OF = 1 - r : r$; $BD : DC = k : 1 - k$;
$AF : FB = n : 1 - n$; $AE : EC = 1 - m : m$
Given the ratio of any two sections, we can find the ratio of the other sections.



Example 1

In the figure, $AF : FB = 2 : 3$, $BD : DC = 4 : 5$
Find $AO : OD$ and $CO : OF$.
Let $AO : OD = 1 - p : p$, $CO : OF = 1 - r : r$
Apply Menelaus' theorem



$$\frac{4+5}{5}\times\frac{p}{1-p}\times\frac{2}{3}=1 \quad \text{(intercept } COF \text{ on } \Delta ABD\text{)}$$

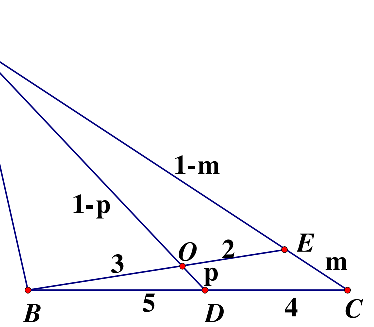
$$\frac{AO}{OD}=\frac{1-p}{p}=\frac{6}{5}$$

$$\frac{4}{5}\times\frac{1-r}{r}\times\frac{2}{2+3}=1 \quad \text{(intercept } AOD \text{ on } \Delta BCF\text{)}$$

$$\frac{CO}{OF}=\frac{1-r}{r}=\frac{25}{8}$$

Example 2

In the figure, $BO : OE = 3 : 2$, $BD : DC = 5 : 4$
Find $AO : OD$ and $AE : EC$.
Let $AO : OD = 1 - p : p$, $AE : EC = 1 - m : m$
Apply Menelaus' theorem



$$\frac{1}{1-m}\times\frac{5}{4}\times\frac{2}{3}=1 \quad \text{(intercept } AOD \text{ on } \Delta BCE\text{)}$$

$$1-m=\frac{5}{6}\Rightarrow m=\frac{1}{6}$$

$$\frac{AE}{EC}=\frac{1-m}{m}=5$$

$$\frac{1-p}{p}\times\frac{5}{5+4}\times\frac{m}{1-m}=1 \quad \text{(intercept } BOE \text{ on } \Delta ACD\text{)}$$

$$\frac{1-p}{p}\times\frac{5}{9}\times\frac{1}{5}=1$$

$$\frac{AO}{OD}=\frac{1-p}{p}=9$$

Example 3

In the figure, $BO : OE = 3 : 2$, $AE : EC = 4 : 5$

Find $AO : OD$ and $BD : DC$.

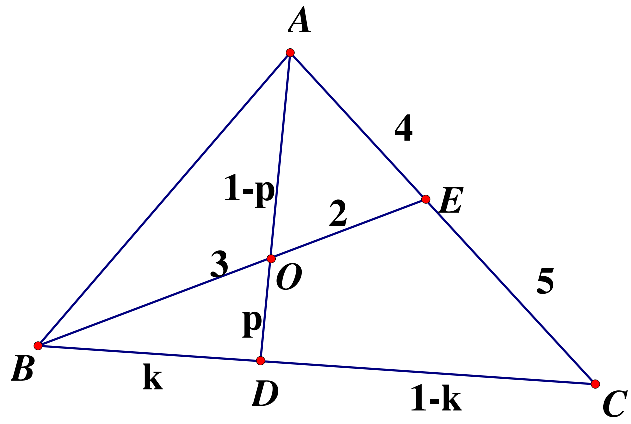
Let $AO : OD = 1 - p : p$, $BD : DC = k : 1 - k$

Apply Menelaus' theorem

$\frac{3}{2}\times\frac{1-k}{k}\times\frac{4}{4+5}=1$ (intercept AOD on ΔBCE)

$\frac{1-k}{k}=\frac{3}{2}$

$\frac{BD}{DC}=\frac{k}{1-k}=\frac{2}{3}$



$\frac{1-p}{p}\times\frac{k}{1}\times\frac{5}{4}=1$ (intercept BOE on ΔACD)

$\frac{1-p}{p}\times\frac{5}{9}\times\frac{1}{5}=1$

$\frac{AO}{OD}=\frac{1-p}{p}=9$

Example 4

In the figure, $BO : OE = 3 : 2$, $AO : OD = 5 : 4$

Find $AE : EC$ and $BD : DC$.

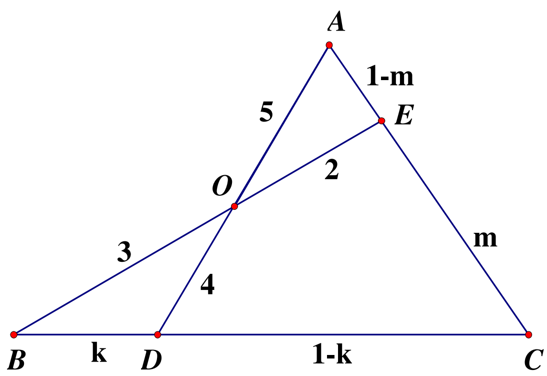
Let $AE : EC = 1 - m : m$, $BD : DC = k : 1 - k$

Apply Menelaus' theorem

$\frac{3}{2}\times\frac{1-k}{k}\times\frac{1-m}{1}=1$ $\cdots$ (1) (intercept AOD on ΔBCE)

$\frac{5}{4}\times\frac{k}{1}\times\frac{m}{1-m}=1$ $\cdots$ (2) (intercept BOE on ΔACD)

(1)×(2): $\frac{15}{8}\times(1-k)\times m=1\Rightarrow(1-k)m=\frac{8}{15}\cdots$ (3)



(1) can be simplified to $\frac{3}{2}\times\frac{1-k-(1-k)m}{k}=1$ $\cdots$ (4)

Sub. (3) into (4): $\frac{1-k-\frac{8}{15}}{k}=\frac{2}{3}\Rightarrow\frac{\frac{7}{15}-k}{k}=\frac{2}{3}\Rightarrow\frac{7-15k}{15k}=\frac{2}{3}\Rightarrow\frac{7-15k}{5k}=2\Rightarrow 7-15k=10k$

$k=\frac{7}{25}\Rightarrow BD : DC = k : 1 - k = 7 : 18$

Sub. $k=\frac{7}{25}$ into (3): $\left(1-\frac{7}{25}\right)m=\frac{8}{15}$

$\Rightarrow m=\frac{20}{27}$

$AE : EC = 1 - m : m = 7 : 20$