Mid-Point Theorem

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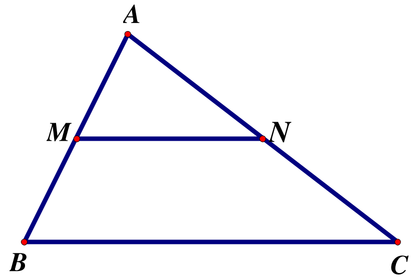
In ΔABC, M is the mid-point of AB, N is the mid-point of AC, then $MN // BC$ and $MN = \frac{1}{2}BC$.

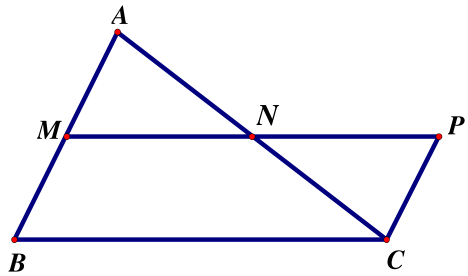


Proof: Method 1

$\frac{AM}{AB} = \frac{1}{2} = \frac{AN}{AC}$	given
$\angle MAN = \angle BAC$	Common angle
$\therefore \Delta AMN \sim \Delta ABC$	2 sides proportional, included angle
$MN = \frac{1}{2}BC$	ratio of sides, ~Δs
$\angle AMN = \angle ABC$	corr. ∠s, ~Δ's
$\therefore MN // BC$	corr. ∠s equal

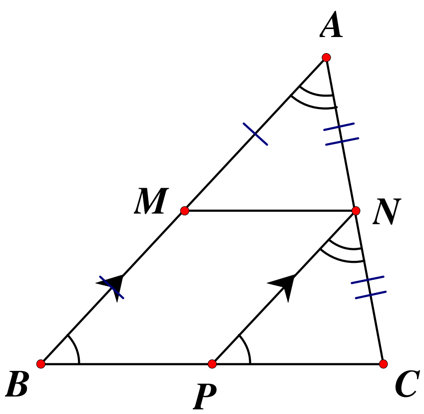
Method 2

Produce MN to P so that $MN = NP$.



$AN = NC$	given
$MN = NP$	By construction
$\angle ANM = \angle CNP$	vert. opp. ∠s
$\therefore \Delta AMN \cong \Delta CPN$	S.A.S.
$CP = AM$	corr. sides, ≅ Δs
$AM = MB$	given
$\therefore MB = PC$	
$\angle MAN = \angle PCN$	corr. ∠s, ≅ Δs
$\therefore AM // PC$	alt. ∠s equal
$MB // PC$	
$BCPM$ is a parallelogram	opp. sides are equal and parallel
$\therefore MNP // BC$	Property of parallelogram
$MN // BC$	
$MN = \frac{1}{2}MP$	by construction
$= \frac{1}{2}BC$	opp. sides of parallelogram
$\therefore MN = \frac{1}{2}BC$	

Method 3



Draw NP // AB, where P lies on BC .

$\angle NCP = \angle ACB$	common
$\angle CPN = \angle CBA$	corr. $\angle$s, NP // AB
$\angle CNP = \angle CAB$	corr. $\angle$s, NP // AB
$\therefore \Delta CNP \neg \Delta CAB$	A.A.A.
$\dfrac{PC}{BC} = \dfrac{NP}{AB} = \dfrac{CN}{CA} = \dfrac{1}{2}$	$\because$ N is the mid-pt. of AC
$\therefore NP = \dfrac{1}{2} AB$ and $PC = \dfrac{1}{2} BC$	…… (*)

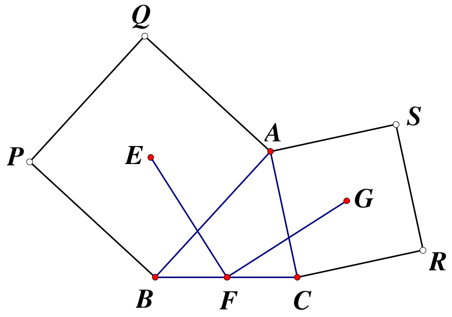
$\because$ M is the mid-pt. of AB and $NP = \dfrac{1}{2} AB$

$\therefore AM = NP = MB$	
$\therefore$ $BPNM$ is a //-gram	opp. sides are eq. and //
MN // BC	property of //-gram
$MN = BP$	opp. sides of //-gram
$\quad = \dfrac{1}{2} BC$	by (*)

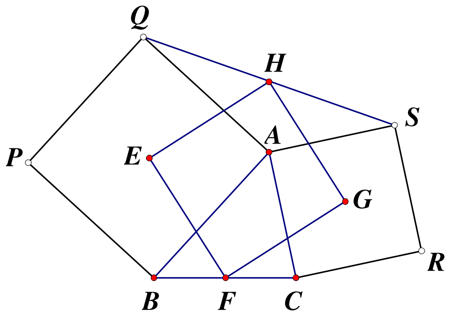
Example on Mid-point Theorem

Modified from HKAL Pure Mathematics 1958 Paper 1 Q6

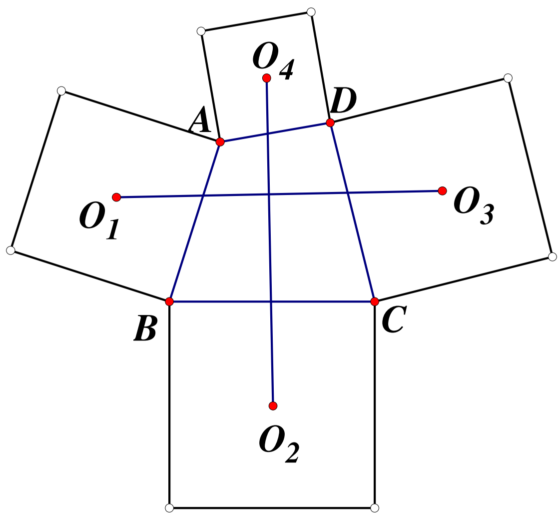
(a) In ΔABC, two squares $ABPQ$ and $ACRS$ are constructed on the sides AB, AC outwards. (See the following figure.) The centres of the squares are E and G respectively. F is the mid-point of BC.
Prove that $EF = FG$ and $EF \perp FG$.



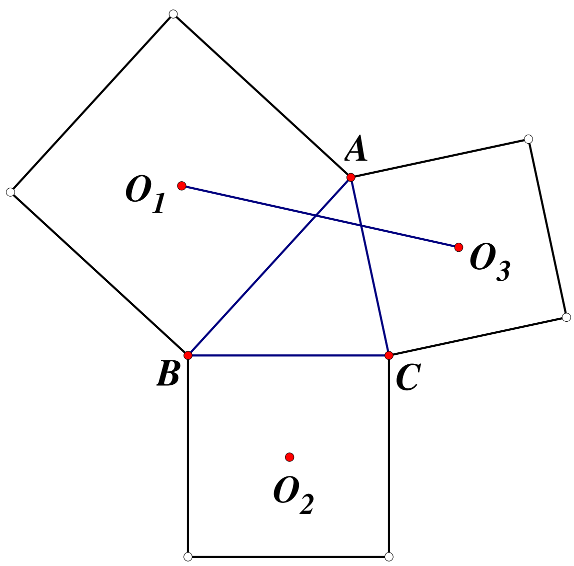
(b) If H is the mid-point of QS, prove that $EFGH$ is a square.

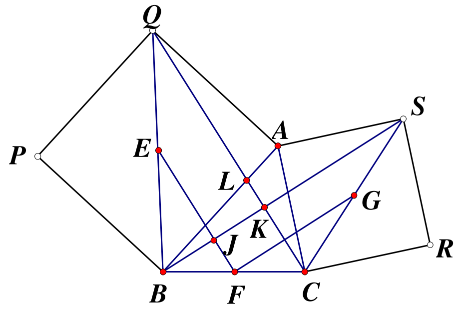


(c) Construct four squares outwards on the sides AB, BC, CD, DA of an arbitrary quadrilateral. (See the following figure.) The centres of the squares are O_1, O_2, O_3, O_4 respectively. Prove that $O_1Q_3 \perp O_2O_4$ and they have equal length.



(d) In part (a), construct one more square outward with side BC, centre O_2.
Which length segment is perpendicular and equal to O_1O_3? Prove your assertion.





(a) In the figure, join the diagonals BQ and CS.

$BE = EQ$ (property of square)

$BF = FC$ (given)

$EF \parallel CQ$ and $EF = \dfrac{CQ}{2} \cdots (1)$ (mid point theorem)

$CG = GS$ (property of square)

$FG \parallel BS$ and $FG = \dfrac{BS}{2} \cdots (2)$ (mid point theorem)

$AQ = AB$ (property of square)

$\angle QAB = \angle CAS = 90°$ (property of square)

$\angle QAC = 90° + \angle BAC = \angle BAC + 90° = \angle BAS$

$AC = AS$ (property of square)

$\therefore \Delta ABQ \cong \Delta APC$ (S.A.S.)

$CQ = BS \cdots (3)$ (corr. sides, $\cong \Delta$s)

$\therefore EF = \dfrac{CQ}{2} = \dfrac{BQ}{2} = FG$ by (1), (2) and (3)

Suppose QC intersects BS and AB at K and L respectively. EF intersects BS at J.

$\angle AQC = \angle ABS$ (corr. $\angle$s $\cong \Delta$s)

$\angle ALQ = \angle BLK$ (vert. opp. $\angle$s)

In ΔBKL,

$\angle BKL = 180° - \angle LBK - \angle BLK$ ($\angle$ sum of Δ)

$= 180° - \angle AQL - \angle ALQ$

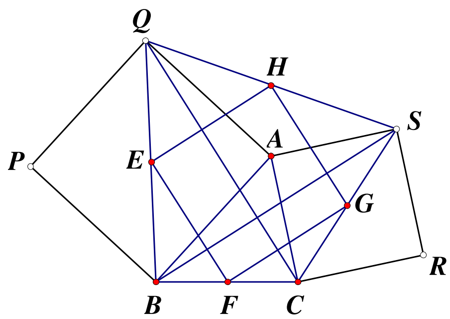
$= \angle QAL$ ($\angle$ sum of Δ)

$= 90°$ (property of square)

$\angle KJE = \angle BKL = 90°$ (int. $\angle$s, $EF \parallel QC$)

$\angle EFG = \angle KJE = 90°$ (corr. $\angle$s, $BS \parallel FG$)

$\therefore EF \perp FG$



(b) $QE = EB$ and $QH = HS$ (given)

$EH \parallel BS \cdots (4)$ (mid-point theorem)

$CF = FB$ and $CG = GS$ (given)

$BS \parallel FG \cdots (5)$ (mid-point theorem)

$\therefore EH \parallel FG \cdots (6)$ (by (4) and (5))

$HG \parallel QC \cdots (7)$ (mid-point theorem)

$QC \parallel EF \cdots (8)$ (mid-point theorem)

$\therefore HG \parallel EF \cdots (9)$ (by (7) and (8))

$EFGH$ is a parallelogram (by (6) and (9))

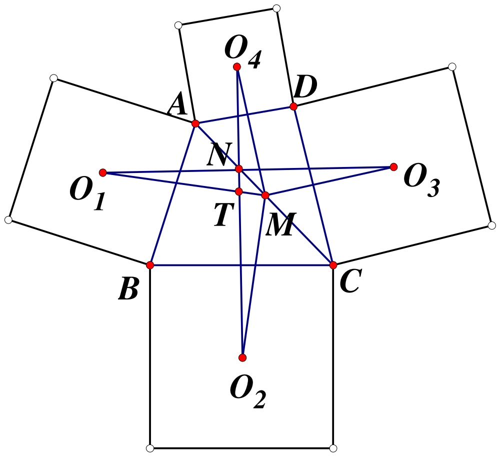
$EF = FG$ and $EF \perp FG$ (by (a))

$\therefore EFGH$ is a square

Example on Mid-point Theorem

Modified from HKAL Pure Mathematics 1958 Paper 1 Q6

(c) Join AC. Let M be the mid-point of AC. Join O_1M, O_2M, O_3M and O_4M.
O_2O_4 meets O_1O_3 and O_1M at N and T respectively.
By the result of (a), $O_1M = O_2M$, $O_3M = O_4M$ and $O_1M \perp O_2M$, $O_3M \perp O_4M$.



$\angle O_1MO_3 = \angle O_1MO_4 + 90°$

$= \angle O_2MO_4$

$\Delta O_1MO_3 \cong \Delta O_2MO_4$ (S.A.S.)

$O_1O_3 = O_2O_4$ (corr. sides, Δs)

$\angle NO_1M = \angle NO_2M \cdots (1)$ (corr. $\angle$s $\cong \Delta$s)

$\angle NTO_1 = \angle MTO_2 \cdots (2)$ (vert. opp. $\angle$s)

$\angle O_1NT = 180° - \angle NO_1T - \angle NTO_1$ ($\angle$ sum of ΔO_1NT)

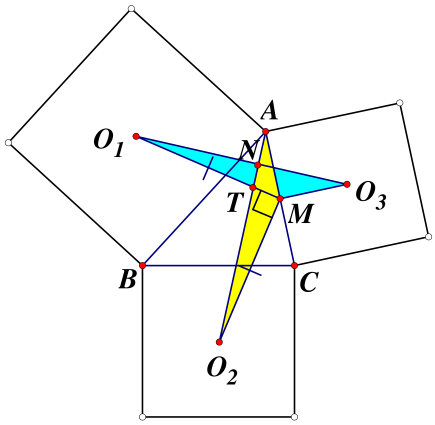
$= 180° - \angle TO_2M - \angle MTO_2$ (by (1) and (2))

$= \angle O_1MO_2$ ($\angle$ sum of ΔO_2MT)

$= 90°$ (proved)

$O_1O_3 \perp O_2O_4$ and they have equal length.

(d) AO_2 is perpendicular and equal to O_1O_3.
Let M be the mid-point of AC.
Join O_1M, O_2M, O_3M, O_1O_3 and AO_2.
O_2A meets O_1O_3 and O_1M at N and T respectively.
By the result of (a), $O_1M = O_2M$ and $O_1M \perp O_2M$.



$\angle O_1MO_3 = \angle AMO_1 + 90°$

$= \angle AMO_2$

$O_3M = AM$ (given)

$\Delta O_1MO_3 \cong \Delta O_2MA$ (S.A.S.)

$O_1O_3 = O_2A$ (corr. sides, Δs)

$\angle NO_1T = \angle MO_2T \cdots (1)$ (corr. $\angle$s $\cong \Delta$s)

$\angle NTO_1 = \angle MTO_2 \cdots (2)$ (vert. opp. $\angle$s)

$\angle O_1NT = 180° - \angle NO_1T - \angle NTO_1$ ($\angle$ sum of ΔO_1NT)

$= 180° - \angle TO_2M - \angle MTO_2$ (by (1) and (2))

$= \angle O_1MO_2$ ($\angle$ sum of ΔO_2MT)

$= 90°$ (proved)

$O_1O_3 \perp O_2A$ and they have equal length.

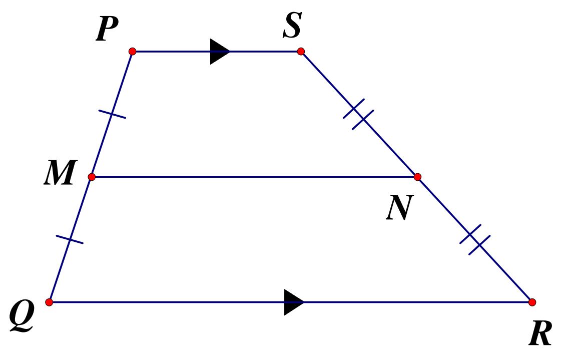
Method 2

By part (c) if $D \to A$ until the square AD diminishes to zero, then $D = A = O_4$.

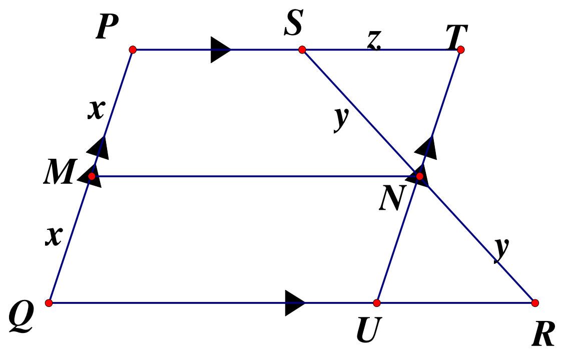
Example 2 on mid-point theorem

In the figure, $PQRS$ is a trapezium with PS // QR.
M is the mid-point of PQ and
N is the mid-point of SR.

Prove that MN // QR and $MN = \frac{1}{2}(PS + QR)$.



Proof: Let $PM = x = MQ$, $SN = y = NR$
Draw a line segment TNU // PMQ, intersecting PS produced at T and QR at U. Let $ST = z$.
By construction, $PQUT$ is a //-gram.
$PT = QU$ and $TU = 2x$ (opp. sides of //-gram)



In ΔSNT and ΔRNU

$\angle SNT = \angle RNU$	(vert. opp. $\angle$s)
$SN = y = NR$	(given)
$\angle NST = \angle NRU$	(alt. $\angle$s, PT // QR)
$\therefore$ $\Delta SNT \cong \Delta RNU$	(A.S.A.)
$TN = NU$	(corr. sides, $\cong$ Δs)
$= \frac{1}{2} TU = \frac{1}{2} \cdot 2x = x$	

$\therefore$ $MNUQ$ is a //-gram (opp. sides are eq. and //)
Also, $PTNM$ is a //-gram (opp. sides are eq. and //)

PT // MN // QU	(property of //-gram)
$PT = MN = QU$	(opp. sides of //-gram)
$ST = UR = z$	(corr. sides, $\cong$ Δs)

$PS = PT - ST = MN - z$

$QR = QU + UR = MN + z$

$PS + QR = MN - z + MN + z = 2MN$

$\therefore$ $MN = \frac{1}{2}(PS + QR)$ Q.E.D.