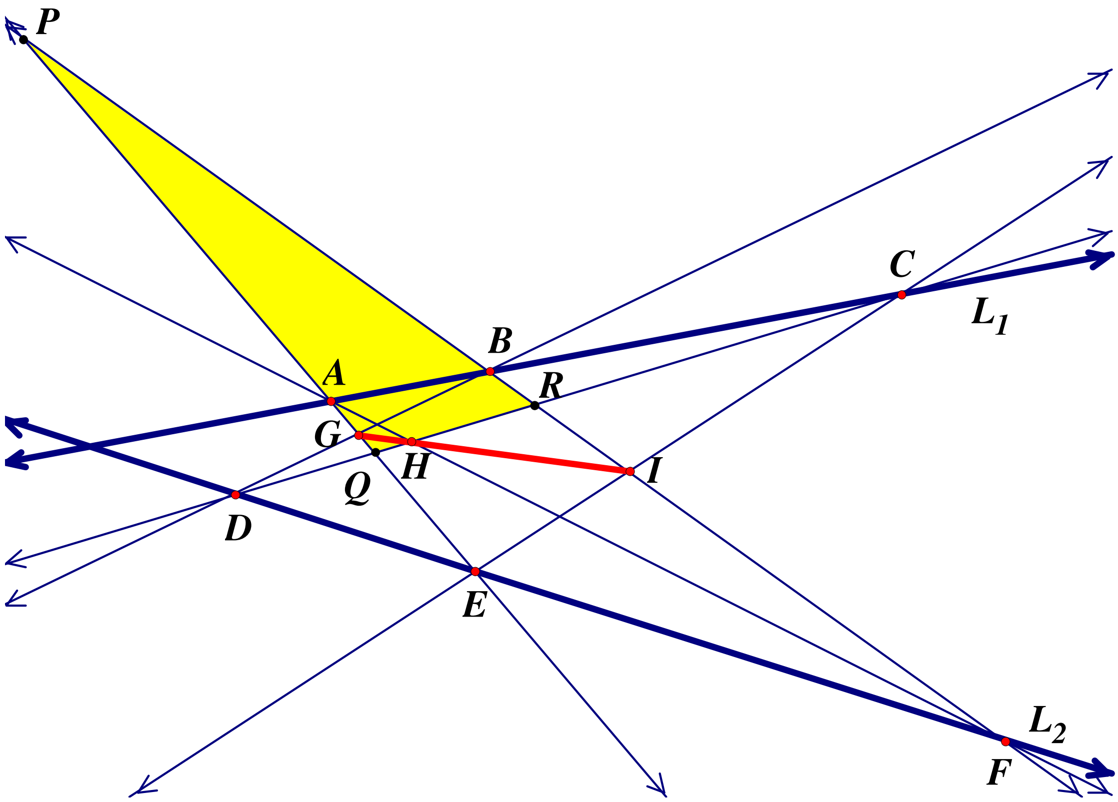
Pappus's theorem

Created by Mr. Francis Hung on 20190116 Last updated: 2021-09-04



A, B and C are three points on a line L_1, D, E and E are three points on another line L_2.
AE and BD intersect at G, AF and CD intersect at H, BF and CE intersect at I.
Prove that G, H and I are collinear.
Proof: Suppose AE is not parallel to BF. They intersect at P. CD intersect AE and BF at Q and R.
Apply Menelaus' theorem on ΔPQR.

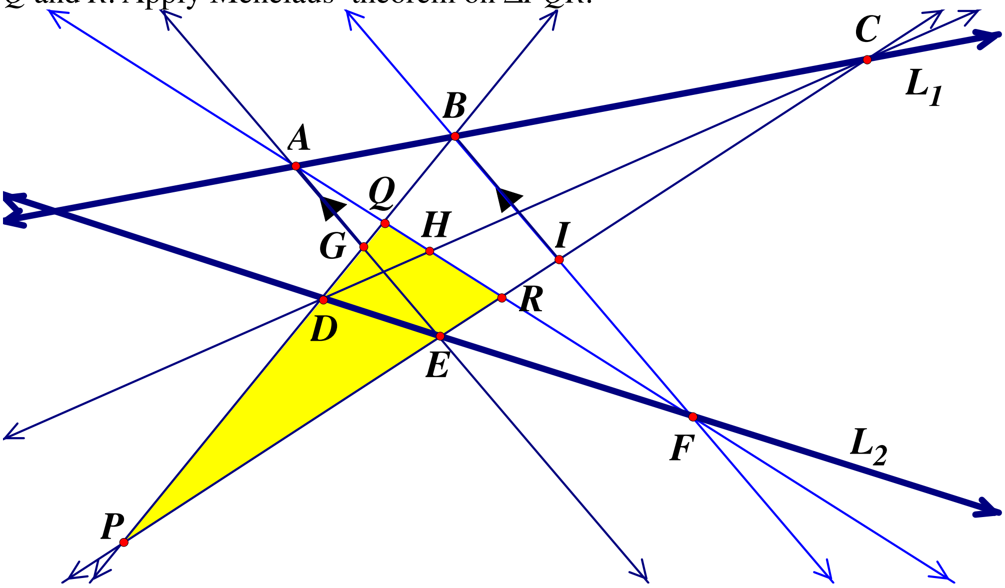
Transversal	Product ratio	Equation
ABC	$\frac{PA}{AQ}\cdot\frac{QC}{CR}\cdot\frac{RB}{BP}=-1$	……(1)
BGD	$\frac{PG}{GQ}\cdot\frac{QD}{DR}\cdot\frac{RB}{BP}=-1$	……(2)
AHF	$\frac{PA}{AQ}\cdot\frac{QH}{HR}\cdot\frac{RF}{FP}=-1$	……(3)
CIE	$\frac{PE}{EQ}\cdot\frac{QC}{CR}\cdot\frac{RI}{IP}=-1$	……(4)
DEF	$\frac{PE}{EQ}\cdot\frac{QD}{DR}\cdot\frac{RF}{FP}=-1$	……(5)

$$\frac{(2)\times(3)\times(4)}{(1)\times(5)}:\frac{\frac{PG}{GQ}\cdot\frac{QD}{DR}\cdot\frac{RB}{BP}\cdot\frac{PA}{AQ}\cdot\frac{QH}{HR}\cdot\frac{RF}{FP}\cdot\frac{PE}{EQ}\cdot\frac{QC}{CR}\cdot\frac{RI}{IP}}{\frac{PA}{AQ}\cdot\frac{QC}{CR}\cdot\frac{RB}{BP}\cdot\frac{PE}{EQ}\cdot\frac{QD}{DR}\cdot\frac{RF}{FP}}=-1$$

$$\frac{PG}{GQ}\cdot\frac{QH}{HR}\cdot\frac{RI}{IP}=-1$$

By the converse of Menelaus' theorem, G, H, I are collinear.

If AE is parallel to BF, but BD is not parallel to CE. They intersect at P. AF intersects BD and CE at Q and R. Apply Menelaus' theorem on ΔPQR.



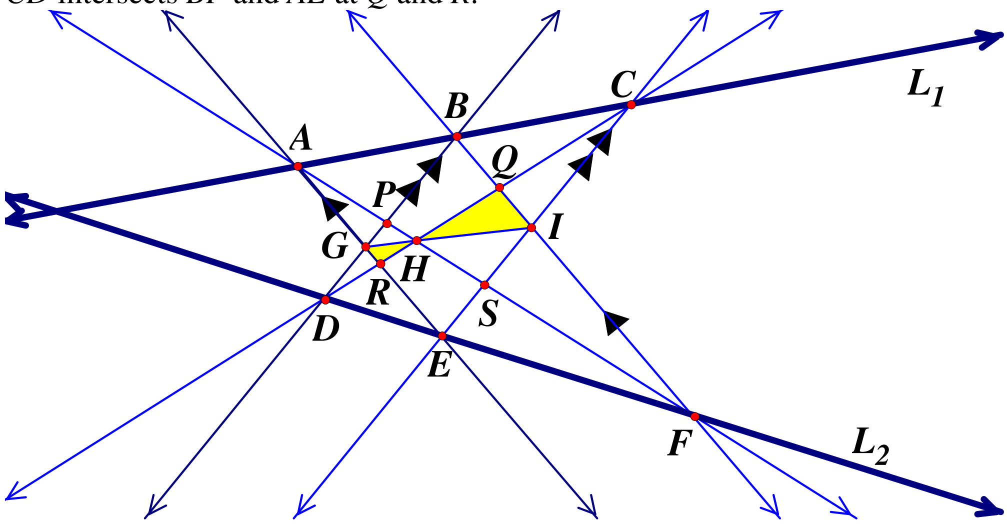
Transversal	Product ratio	Equation
ABC	$\frac{PB}{BQ}\cdot\frac{QA}{AR}\cdot\frac{RC}{CP}=-1$	……(1)
BIF	$\frac{PB}{BQ}\cdot\frac{QF}{FR}\cdot\frac{RI}{IP}=-1$	……(2)
AGE	$\frac{PG}{GQ}\cdot\frac{QA}{AR}\cdot\frac{RE}{EP}=-1$	……(3)
CHD	$\frac{PD}{DQ}\cdot\frac{QH}{HR}\cdot\frac{RC}{CP}=-1$	……(4)
DEF	$\frac{PD}{DQ}\cdot\frac{QF}{FR}\cdot\frac{RE}{EP}=-1$	……(5)

$$\frac{(2)\times(3)\times(4)}{(1)\times(5)}:\frac{\frac{PB}{BQ}\cdot\frac{QF}{FR}\cdot\frac{RI}{IP}\cdot\frac{PG}{GQ}\cdot\frac{QA}{AR}\cdot\frac{RE}{EP}\cdot\frac{PD}{DQ}\cdot\frac{QH}{HR}\cdot\frac{RC}{CP}}{\frac{PB}{BQ}\cdot\frac{QA}{AR}\cdot\frac{RC}{CP}\cdot\frac{PD}{DQ}\cdot\frac{QF}{FR}\cdot\frac{RE}{EP}}=-1$$

$$\frac{PG}{GQ}\cdot\frac{QH}{HR}\cdot\frac{RI}{IP}=-1$$

By the converse of Menelaus' theorem, G, H, I are collinear.

If AE is parallel to BF and BD is parallel to CE. Suppose AF intersects BD and CE at P and S, CD intersects BF and AE at Q and R.



We can use the properties of similar triangles and the ratio of the corresponding sides to derive the results:

Similar triangles	Ratios of corresponding sides	Equation
$\Delta DGR \sim \Delta DBQ$	$\frac{GR}{BQ}=\frac{DR}{DQ}$	……(1)
$\Delta DER \sim \Delta DFQ$	$\frac{ER}{FQ}=\frac{DR}{DQ}$	……(2)
$\Delta CBQ \sim \Delta CAR$	$\frac{BQ}{AR}=\frac{CQ}{CR}$	……(3)
$\Delta CIQ \sim \Delta CER$	$\frac{IQ}{ER}=\frac{CQ}{CR}$	……(4)
$\Delta AHR \sim \Delta FQH$	$\frac{RH}{HQ}=\frac{AR}{FQ}$	……(5)

$$\frac{(2)\times(4)}{(1)\times(3)}:\ \frac{\frac{ER}{FQ}\cdot\frac{IQ}{ER}}{\frac{GR}{BQ}\cdot\frac{BQ}{AR}}=\frac{\frac{DR}{DQ}\cdot\frac{CQ}{CR}}{\frac{DR}{DQ}\cdot\frac{CQ}{CR}}=1$$

$$\frac{AR}{FQ}\cdot\frac{IQ}{GR}=1\Rightarrow\frac{AR}{FQ}=\frac{GR}{IQ}\ \cdots\cdots(6)$$

Compare (5) and (6), we have $\frac{RH}{HQ}=\frac{GR}{IQ}$

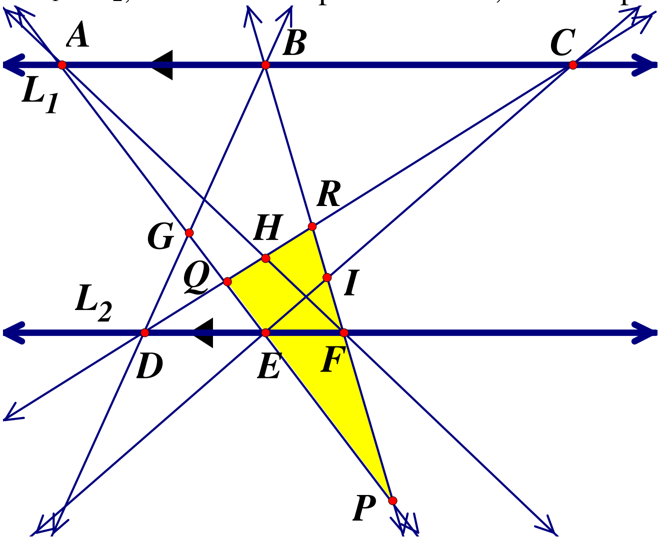
$\angle GRH = \angle IQH$ (alt. $\angle$s AE // BF)

$\therefore\ \Delta GRH \sim \Delta IQH$ (2 sides proportional, included $\angle$s)

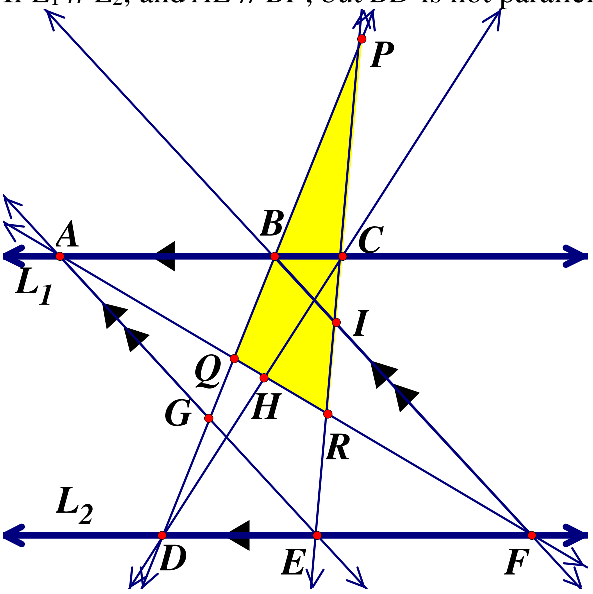
$\angle GHR = \angle IHQ$ (corr. $\angle$s, $\sim\Delta$s)

$\therefore$ G, H, I are collinear (vert. opp. $\angle$s equal and QHR is a st. line)

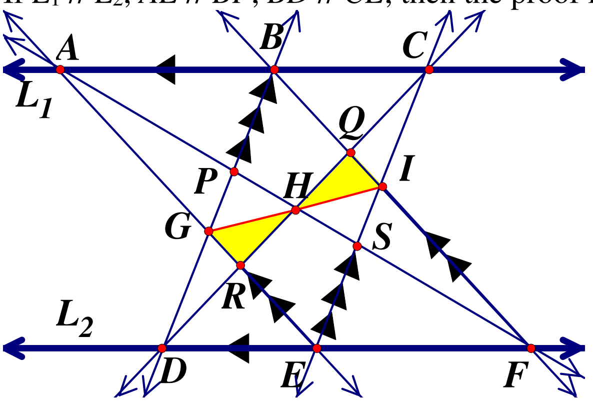
If L_1 // L_2, and AE is not parallel to BF, then the proof is similar and so is omitted.



If L_1 // L_2, and AE // BF, but BD is not parallel to CE, then the proof is similar and so is omitted.

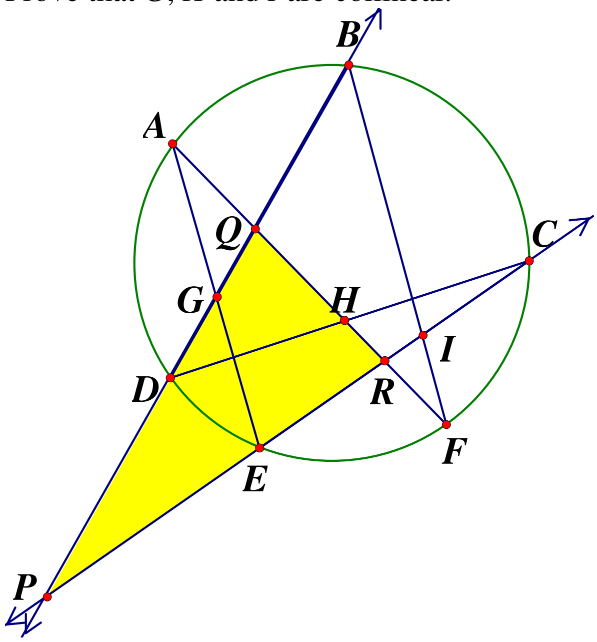


If L_1 // L_2, AE // BF, BD // CE, then the proof is similar and so is omitted.



Pascal's theorem

A, *B*, *C*, *D*, *E, F* are six points in order on the circumference of a circle.
AE and *BD* intersect at *G*, *AF* and *CD* intersect at *H*, *BF* and *CE* intersect at *I*.
Prove that *G*, *H* and *I* are collinear.



Proof: Suppose *BD* is not parallel to *CE*. They intersect at *P*. *AF* intersects *BD* and *CE* at *Q* and *R*.
By intersecting chords theorem,

$$PB\times PD = CP\times EP \Rightarrow \frac{PB}{CP}\cdot\frac{PD}{EP}=1 \quad \cdots\cdots(1)$$

$$BQ\times DQ = QA\times QF \Rightarrow \frac{QA}{BQ}\cdot\frac{QF}{DQ}=1 \cdots\cdots(2)$$

$$AR\times FR = RC\times RE \Rightarrow \frac{RC}{AR}\cdot\frac{RE}{FR}=1 \quad \cdots\cdots(3)$$

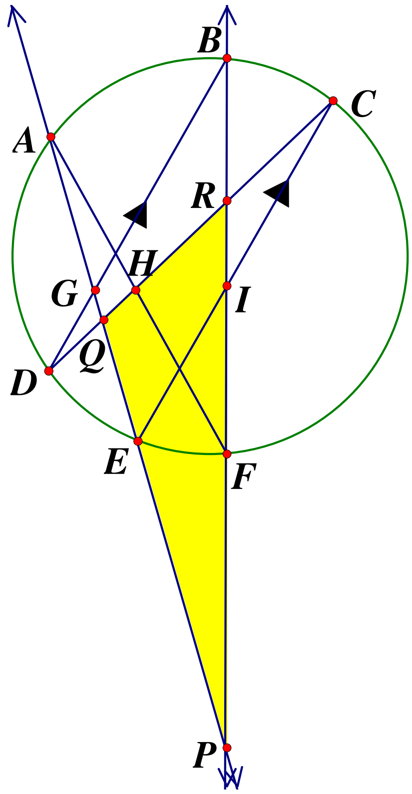
Apply Menelaus' theorem on ΔPQR:

Transversal	Product ratio	Equation
BIF	$\frac{PB}{BQ}\cdot\frac{QF}{FR}\cdot\frac{RI}{IP}=-1$	$\cdots\cdots(4)$
AGE	$\frac{PG}{GQ}\cdot\frac{QA}{AR}\cdot\frac{RE}{EP}=-1$	$\cdots\cdots(5)$
CHD	$\frac{PD}{DQ}\cdot\frac{QH}{HR}\cdot\frac{RC}{CP}=-1$	$\cdots\cdots(6)$

$$\frac{(4)\times(5)\times(6)}{(1)\times(2)\times(3)}:\frac{\frac{PB}{BQ}\cdot\frac{QF}{FR}\cdot\frac{RI}{IP}\cdot\frac{PG}{GQ}\cdot\frac{QA}{AR}\cdot\frac{RE}{EP}\cdot\frac{PD}{DQ}\cdot\frac{QH}{HR}\cdot\frac{RC}{CP}}{\frac{PB}{CP}\cdot\frac{QA}{BQ}\cdot\frac{RC}{AR}\cdot\frac{PD}{EP}\cdot\frac{QF}{DQ}\cdot\frac{RE}{FR}}=-1$$

$$\frac{PG}{GQ}\cdot\frac{QH}{HR}\cdot\frac{RI}{IP}=-1$$

By the converse of Menelaus' theorem, *G*, *H*, *I* are collinear.



If BD is parallel to CE, but AE is not parallel to BF. They intersect at P. CD intersects AE and BF at Q and R.
By intersecting chords theorem,

$$PA\times PE = PB\times PF \Rightarrow \frac{PB}{CP}\cdot\frac{PD}{EP}=1 \quad \cdots\cdots(1)$$

$$BR\times FR = CR\times DR \Rightarrow \frac{QA}{BQ}\cdot\frac{QF}{DQ}=1 \quad \cdots\cdots(2)$$

$$AQ\times QE = CQ\times QD \Rightarrow \frac{RC}{AR}\cdot\frac{RE}{FR}=1 \quad \cdots\cdots(3)$$

Apply Menelaus' theorem on ΔPQR.

Transversal	Product ratio	Equation
CIE	$\frac{PI}{IR}\cdot\frac{RC}{CQ}\cdot\frac{QE}{EP}=-1$	$\cdots\cdots(4)$
AHF	$\frac{PA}{AQ}\cdot\frac{QH}{HF}\cdot\frac{RF}{FP}=-1$	$\cdots\cdots(5)$
BGD	$\frac{PG}{GQ}\cdot\frac{QD}{DR}\cdot\frac{RB}{BP}=-1$	$\cdots\cdots(6)$

$$\frac{(4)\times(5)\times(6)}{(1)\times(2)\times(3)}:\frac{\frac{PI}{IR}\cdot\frac{RC}{CQ}\cdot\frac{QE}{EP}\cdot\frac{PA}{AQ}\cdot\frac{QH}{HF}\cdot\frac{RF}{FP}\cdot\frac{PG}{GQ}\cdot\frac{QD}{DR}\cdot\frac{RB}{BP}}{\frac{PB}{CP}\cdot\frac{QA}{BQ}\cdot\frac{RC}{AR}\cdot\frac{PD}{EP}\cdot\frac{QF}{DQ}\cdot\frac{RE}{FR}}=-1$$

By the converse of Menelaus' theorem, G, H, I are collinear.