Theorem of equal ratios

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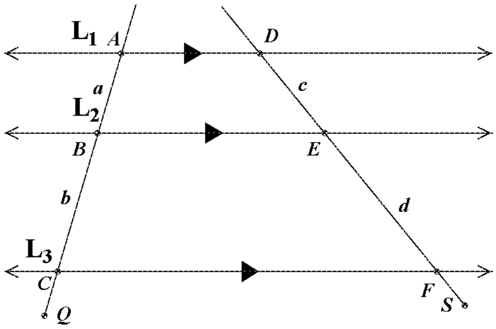
Given three parallel lines L_1, L_2, L_3.

Two intercepts PQ and RS intersecting L_1, L_2, L_3 at A, B, C, D, E, F respectively as shown in the figure.

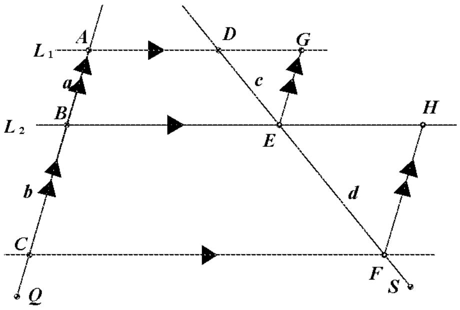


Then $\frac{AB}{BC} = \frac{DE}{EF}$.

Proof: Suppose PQ and RS are not parallel

Let $AB = a$, $BC = b$, $DE = c$, $EF = d$

Draw DG and FH parallel to ABC, cutting AD produced at G and BE produced at H



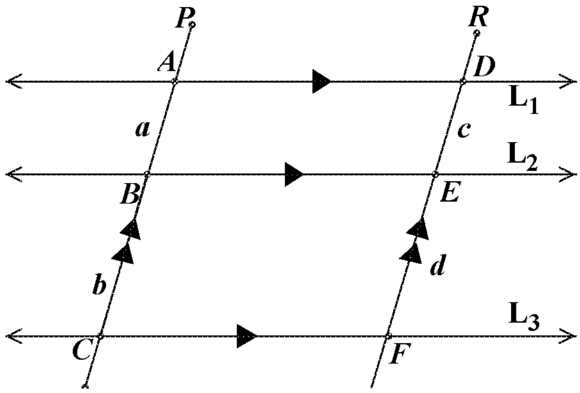
$ABEG$ and $BCFH$ are parallelograms

$GE = AB = a$, $HF = BC = b$	(opp. sides of //-gram)
$\angle EDG = \angle FEH$	(corr. $\angle$s, AG // BH)
$\angle EGD = \angle GEH$	(alt. $\angle$s, DG // EH)
$\angle GEH = \angle EHF$	(alt. $\angle$s, GE // HF)
$\therefore \angle EGD = \angle EHF$	
$\angle DEG = \angle EFH$	(corr. $\angle$s, EG // FH)
$\Delta DEG \sim \Delta EFH$	(equiangular)
$\frac{a}{b} = \frac{c}{d}$	(corr. sides, ~Δs)
$\frac{AB}{BC} = \frac{DE}{EF}$	

If PQ // RS, then $ABED$ and $BCFE$ are parallelograms.

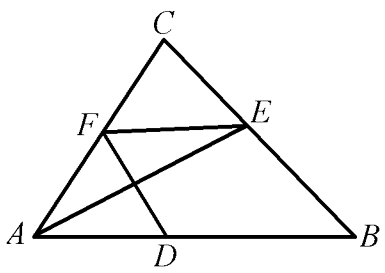
$AB = DE$ and $BC = EF$ (opp. sides, //-gram)

$\frac{AB}{BC} = \frac{DE}{EF}$



Example 1

As shown in the figure, the area of ΔABC is 10. D, E, F are points on AB, BC and CA respectively such that $AD : DB = 2 : 3$, and area of ΔABE = area of quadrilateral $BEFD$. Find the area of ΔABE.



Join DE. Area of ΔADE = area of ΔDEF

$\because$ ΔADE and ΔDEF have the same base and the same height

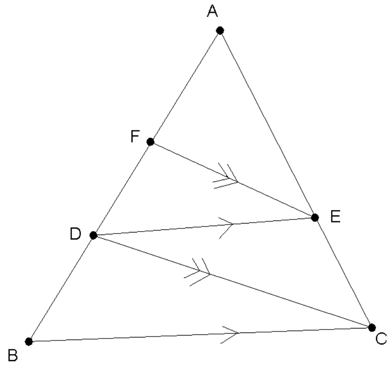
$\therefore$ $DE // AC$

$BE : EC = BD : DA = 3 : 2$ (theorem of equal ratio)

Area of ΔABE = Area of $\Delta ABC \times \frac{BE}{BC} = 10 \times \frac{3}{3+2} = 6$

Example 2

In ΔABC, $DE // BC$, $FE // DC$, $AF = 2$, $FD = 3$ and $DB = X$. Find the value of X.



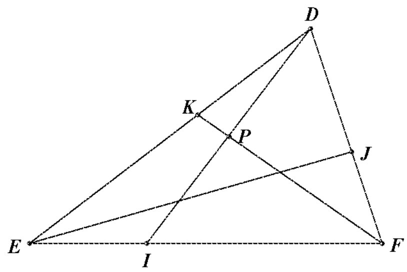
$AE : EC = 2 : 3$ (theorem of eq. ratio)

$AD : DB = 2 : 3$ (theorem of eq. ratio)

$DB = (2+3) \times \frac{3}{2} = 7.5$

Example 3

In Figure 5, the area of ΔDEF is 30 cm^2. EIF, DJF and DKE are straight lines. P is the intersection point of DI and FK. Let $EI : IF = 1: 2$, $FJ : JD = 3 : 4$, $DK : KE = 2 : 3$.

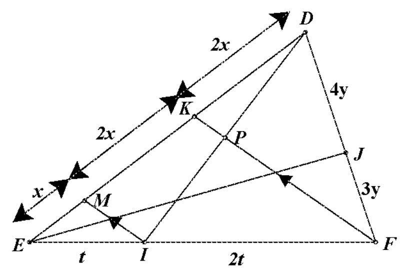


Let the area of ΔDFP be B cm^2, find the value of B.

Let $EI = t$, $IF = 2t$, $DK = 2x$, $KE = 3x$

Draw a line IM parallel to KF cutting DE at M.

By the theorem of equal ratio, $\frac{EM}{MK} = \frac{EI}{IF} = \frac{1}{2}$



$\therefore$ $EM = x$, $MK = 2x$

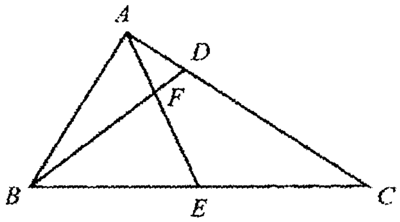
$DP = PI$ (Intercept theorem)

Area of $\Delta DIF = \frac{2}{3} \cdot$ area of ΔDFE = 20 cm^2

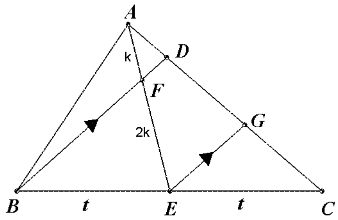
Area of $\Delta DFP = \frac{1}{2} \cdot$ area of ΔDIF = 10 cm^2

$\Rightarrow B = 10$

Example 4



In the figure, ABC is a triangle, E is the midpoint of BC, F is a point on AE where $AE = 3AF$. The extension segment of BF meets AC at D. Given that the area of ΔABC is 48 cm^2. Let the area of ΔAFD be g cm^2, find the value of g.



From E, draw a line EG // BD which cuts AC at G.

$AE = 3AF \Rightarrow AF : FE = 1 : 2$; let $AE = k$, $FE = 2k$

E is the mid-point of $BC \Rightarrow BE = EC = t$

$S_{ABE} = S_{ACE} = \frac{1}{2}\cdot 48$ cm^2 = 24 cm^2 (same base, same height)

$AD : DG = AF : FE = 1 : 2$ (theorem of equal ratio)

$DG : GC = BE : EC = 1 : 1$ (theorem of equal ratio)

$\therefore AD : DG : GC = 1 : 2 : 2$

$S_{AEG} : S_{CEG} = 3 : 2$ (same height, ratio of base = 3 : 2)

$S_{AEG} = 24\times\frac{3}{2+3}\text{cm}^2 = \frac{72}{5}\text{cm}^2$

$\Delta ADF \sim \Delta AGE$

$\Rightarrow S_{ADF} = \frac{1}{9}S_{AEG} = \frac{1}{9}\cdot\frac{72}{5}\text{cm}^2 = \frac{8}{5}\text{cm}^2$

$g = \frac{8}{5}$