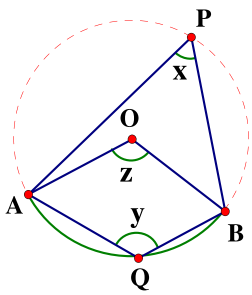
Angles in a circle

Created by Mr. Francis Hung on 20210922 Last updated: 2021-09-22



In the figure, A, P, B, Q are 4 points (in order) in a circle, centre at O.

$\angle AOB = z$ is called the angle at centre subtended by the arc $\overset{\frown}{AQB}$ or $\overset{\frown}{AB}$.

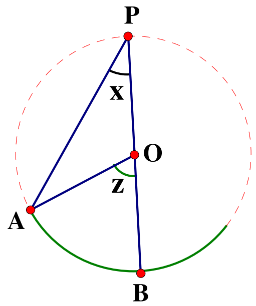
Reflex $\angle AOB = 360° - z$ ($\angle$s at a point)

This is the angle at centre subtended by the arc $\overset{\frown}{APB}$.

$\angle APB = x$ is called the angle at circumference (or $\angle$ at $\odot^{ce}$) subtended by the arc $\overset{\frown}{AQB}$ or $\overset{\frown}{AB}$.

$\angle AQB = y$ is the angle at circumference (or $\angle$ at $\odot^{ce}$) subtended by arc $\overset{\frown}{APB}$.

Theorem 1 Angle at centre twice angle at at circumference $\angle AOB = 2\angle APB$. i.e. $z = 2x$



Case 1 P, O, A are collinear or P, O, B are collinear.

Without loss of generality, assume P, O, B are collinear.

$\angle APB = x$, $\angle AOB = z$

$OA = OP$ (radii)

$\therefore$ ΔOAP is an isosceles Δ

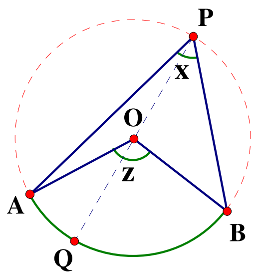
$\angle OAP = \angle APB = x$ (base $\angle$s isos. Δ)

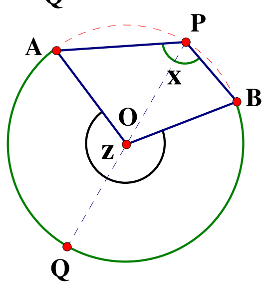
In ΔOAP,

$\angle OAP + \angle APB = \angle AOB$ (ext. $\angle$ of Δ)

$x + x = z$

$\therefore$ $z = 2x$





Case 2 P, O, A are not collinear and P, O, B are not collinear

AP does not intersect OB and BP does not intersect OA.

$\angle APB = x$, $\angle AOB = z$

Join PO and produce it to meet the circle again at Q.

$OP = OA = OB$ (radii)

ΔOAP and ΔOBP are isosceles Δs

$\angle OAP = \angle APO$ (base $\angle$s isos. Δ)

$\angle OBP = \angle BPO$ (base $\angle$s isos. Δ)

In ΔOAP,

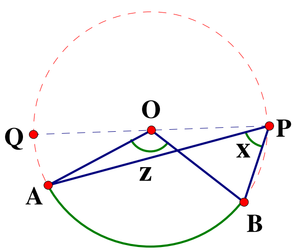
$\angle OAP + \angle APO = \angle AOQ$ (ext. $\angle$ of Δ) $\cdots\cdots$ (1)

In ΔOBP,

$\angle OBP + \angle BPO = \angle BOQ$ (ext. $\angle$ of Δ) $\cdots\cdots$ (2)

(1) + (2): $x + x = z$

$\therefore$ $z = 2x$



Case 3 P, O, A are not collinear and P,O, B are not collinear

AP intersects OB or BP intersects OA.

Without loss of generality, assume AP intersects OB

$\angle APB = x$, $\angle AOB = z$

Join PO and produce it to meet the circle again at Q.

$OP = OA = OB$ (radii)

ΔOAP and ΔOBP are isosceles Δs

$\angle OAP = \angle APO$ (base $\angle$s isos. Δ)

$\angle OBP = \angle BPO$ (base $\angle$s isos. Δ)

In ΔOAP,

$\angle OAP + \angle APO = \angle AOQ$ (ext. $\angle$ of Δ) $\cdots\cdots$ (3)

In ΔOBP,

$\angle OBP + \angle BPO = \angle BOQ$ (ext. $\angle$ of Δ) $\cdots\cdots$ (4)

(4) – (3): $x + x = z$

$\therefore$ $z = 2x$

Abbreviation:

$\angle$ at centre twice $\angle$ at $\odot^{ce}$

Angle in semi-circle

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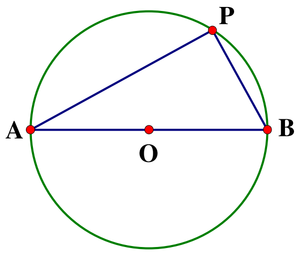


In the figure, *AB* is a diameter in a circle, centre at *O*. *P* is any point on the circle other than *A* and *B*. Then $\angle APB = 90°$.

Proof: $\angle AOB = 180°$ (adj. ∠s on st. line)

$\angle AOB = 2\angle APB$ (∠ at centre twice ∠ at ⊙^{ce})

$\angle APB = 90°$

Method 2 Join *OP*.

$OA = OP = OB$ (radii)

ΔOAP and ΔOBP are isosceles Δs

$\angle OAP = \angle APO$ (base ∠s isos. Δ)

$\angle OBP = \angle BPO$ (base ∠s isos. Δ)

In ΔOAB,

$\angle BAP + \angle ABP + \angle APB = 180°$ (∠ sum of Δ)

$2\angle APO + 2\angle BPO = 180°$

$\angle APO + \angle BPO = 90°$

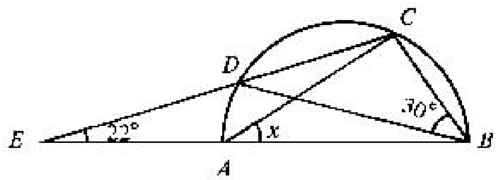
$\angle APB = 90°$

Abbreviation:

∠ in semi-circle

Example 1 (HKCEE 1994 Paper 2 Q51)

In the figure, *ABCD* is a semi-circle, *CDE* and *BAE* are straight lines. If $\angle CBD = 30°$ and $\angle DEA = 22°$, find *x*.



In ΔACE, $\angle ACE = x - 22°$ (ext. ∠ of Δ)

$\angle ABD = x - 22°$ (∠s in the same segment)

$\angle ACB = 90°$ (∠ in semi-circle)

$22° + 2(x - 22°) + 90° + 30° = 180°$ (∠ sum of Δ)

$x = 41°$

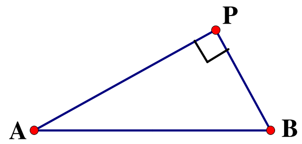
Converse, angle in semi-circle

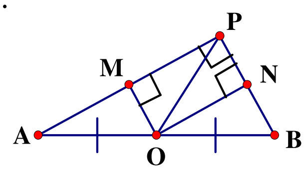
In the figure, *ABP* is a right-angled traingle with $\angle APB = 90°$.

Then we can draw a circle to pass through *A*¸*B* and *P* with *AB* as diameter.

Let *O* be the mid-point of *AB*. Join *OP*.

Let *M* and *N* be the feet of perpendiculars from *O* to *AB* and *BP* respectively.





$\angle MON = 90°$ (∠ sum of polygon)

O, *M*, *N*, *P* is a rectangle

$OM = NP$, $ON = MP$ (opp. sides of rectangle)

$\Delta POM \cong \Delta OPN$ (R.H.S.)

OM // *BP*, *ON* // *AP* (int. ∠s supp.)

$AM = MP$, $BN = NP$ (intercept theorem)

$\Delta AOM \cong \Delta POM$, $\Delta OPN \cong \Delta OBM$ (S.A.S.)

∴ $\Delta AOM \cong \Delta POM \cong \Delta OPN \cong \Delta OBM$

$OA = OP = OB$ (cor. sides ≅Δs)

∴ We can draw a circle to pass through *A*¸*B* and *P* with *O* as centre, i.e. *AB* is a diameter.

Angles in the same segment

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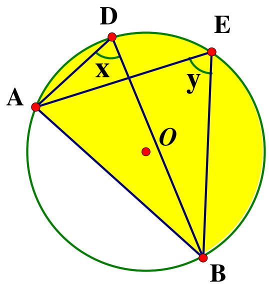


In the figure, AB is a chord in a circle, centre at O. D and E are any points on the same arc $\overset{\frown}{AB}$ (either major arc or minor arc). Then $\angle ADB = \angle AEB$.

Let $\angle ADB = x$, $\angle AEB = y$

Join OA, OB.

$\angle AOB = 2x$ (∠ at cnetre twice ∠ at ⊙^{ce})

$\angle AOB = 2y$ (∠ at cnetre twice ∠ at ⊙^{ce})

$\therefore 2x = 2y$

$x = y$

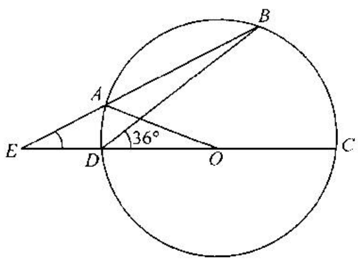
$\angle ADB = \angle AEB$

The theorem is proved.

Abbreviation:
∠s in the same seg.

Example 2 (HKCEE 2004 Paper 2 Q23)

In the figure, O is the centre of the circle $ABCD$. If EAB and $EDOC$ are straight lines and $EA = AO$, find $\angle AEO$.



Let $\angle ABD = x$

$\angle AOE = 2x$ (∠ at centre = 2∠ at ⊙^{ce})

$\angle AEO = 2x$ (base ∠s isos. Δ)

$x + 2x = 36°$ (ext. ∠ of Δ)

$x = 12°$

$\angle AEO = 24°$