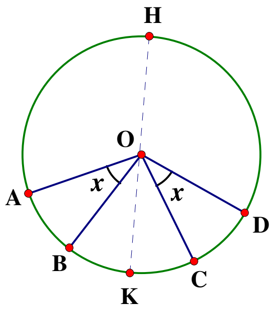
Chords and arcs

Created by Francis Hung on 20210921   Last updated: 26 September 2021



The figure represents two minor arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ of a circle passing through A, B, C, D with centre at O, such that $\angle AOB = \angle COD = x$.
If we draw the diameter HOK which bisects $\angle BOC$, it also bisects $\angle AOD$. But the circle is symmetrical about the diameter HOK, therefore, if we fold the figure about HOK, B can be made to coincide with C, and A with D, and the minor arc $\overset{\frown}{AB}$ will coincide with the minor arc $\overset{\frown}{CD}$.

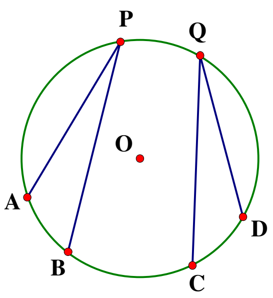
i.e. we have proved: equal angles at centre subtend equal arcs.

Abbreviation: **eq. ∠s, eq. arcs.**

Conversely, if the arcs $\overset{\frown}{AB} = \overset{\frown}{CD}$, then we can again fold the figure about HOK, the line of symmetry about $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$. Then the line OB will coincide with OC, and that the line OA will coincide with OD. Therefore, the two angles at centre are equal: $\angle AOB = \angle COD$.

i.e. we have proved: equal arcs subtend equal angles at centre.

Abbreviation: **eq. arcs, eq. ∠s**



Theorem Equal angles at circumference subtend equal arcs.

In the figure, $\angle APB = \angle CQD$, then $\overset{\frown}{AB} = \overset{\frown}{CD}$.

Proof: Let O be the centre, join OA, OB, OC and OD.

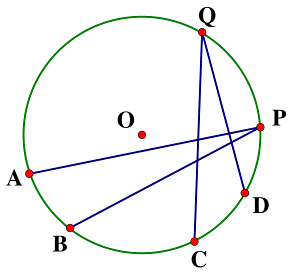
Then $\angle APB = \angle CQD$ (given)

$2\angle APB = 2\angle CQD$

$\angle AOB = \angle COD$ (∠ at centre twice ∠ at ⊙[ce])

$\therefore \overset{\frown}{AB} = \overset{\frown}{CD}$ (eq. ∠s eq. arcs)

Abbreviation: **eq. ∠s eq. arcs**

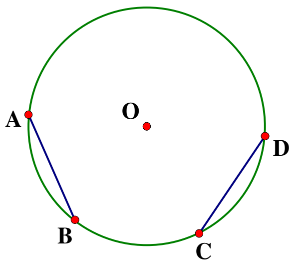


Theorem Equal arcs subtend equal angles at circumference.

In the figure, $\overset{\frown}{AB} = \overset{\frown}{CD}$, then $\angle APB = \angle CQD$.

Proof: Exercise

Abbreviation: **eq. arcs eq. ∠s**



Theorem If $\overset{\frown}{AB} = \overset{\frown}{CD}$, then $AB = CD$.

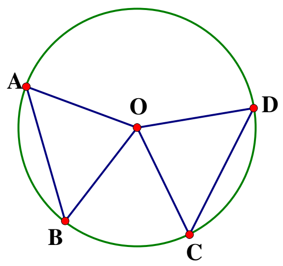
Abbreviation: **eq. arcs, eq. chords.**

Proof: Let O be the centre, join OA, OB, OC and OD.

$\Delta OAB \cong \Delta OCD$ (S.A.S.)

Conversely, if = , then =

Abbreviation: eq. chords, eq. arcs.

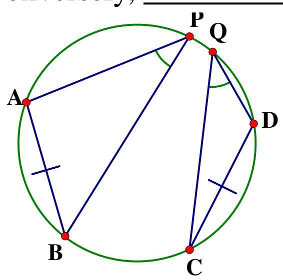


Theorem If AB and CD are equal chords and O is the centre, then $\angle AOB = \angle COD$.

Abbreviation: **eq. chords eq. $\angle$s.**

Proof: exercise.

Conversely, ______________________________



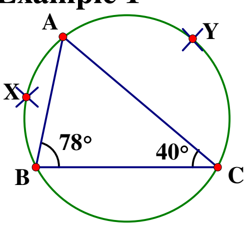
Theorem Equal chords subtend equal angles at circumference.

If $AB = CD$, P and Q are any points on the circumference, then $\angle APB = \angle CQD$.

Abbreviation: **eq. chords eq. $\angle$s.**

Proof: exercise.

Example 1



X and Y are mid-points of the arcs $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$ respectively, find $\angle XAC$ and $\angle BXY$.

$\angle XAB = \frac{1}{2}\angle ACB = 20°$ ($\angle \propto$ arcs)

$\angle BAC = 180° - 78° - 40° = 62°$ ($\angle$ sum of Δ)

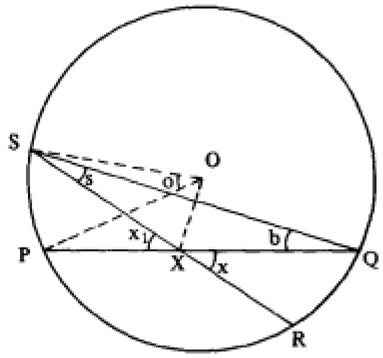
$\angle XAC = \angle XAB + \angle BAC = 20° + 62° = 82°$

$\angle BXC = \angle BAC = 62°$ ($\angle$s in the same segment)

$\angle CXY = \frac{1}{2}\angle CBA = 39°$ ($\angle \propto$ arcs)

$\angle BXY = \angle BXC + \angle CXY = 62° + 39° = 101°$

Example 2



In the figure, 2 chords PQ, RS intersect at X, $\overset{\frown}{PS} = \overset{\frown}{QR}$, O is the centre of the circle. To prove

(a) $\angle QXR = 2\angle SQX$,

(b) P, S, O, X are concyclic

(a) Using the notation in the figure,

$x = x_1$ (vert. opp. ∠s)

$b + s = x_1$ (ext. ∠ of Δ)

$b = s$ (eq. arcs eq. ∠s)

$\therefore x = b + s = 2b$

$\angle QXR = 2\angle SQX$

(b) $o = 2b$ (∠ at centre twice ∠ at ⊙[ce])

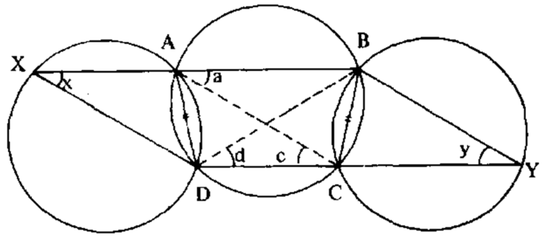
$x = 2b$ (proved in (a))

$\therefore o = x$

P, S, O, X are concyclic (converse, ∠s in the same seg.)

Example 3

In the figure, three equal circles cutting A, D and B, C respectively such that chords $AD = BC$, XAB and DCY are straight lines. To prove $XBYD$ is a //-gram.



Join AC, DB. Let $\angle BAC = a$, $\angle ACD = c$, $\angle BDC = d$, $\angle AXD = x$, $\angle BYC = y$.

$a = c$ (eq. chords eq. ∠s)

XAB // DCY (alt. ∠s eq.)

$\angle ABD = x = d = y$ (eq. chords eq. ∠s)

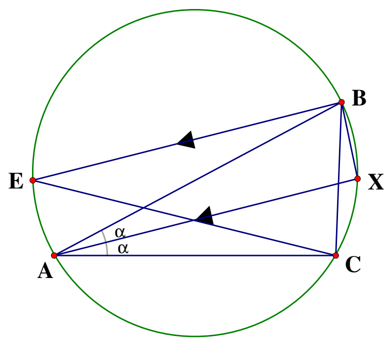
$\angle BDX = 180° - 2x = 180° - 2y = \angle YBD$ (∠ sum of Δ)

XD // BY (alt. ∠ eq.)

$\therefore$ $XDBY$ is a //-gram (opp. sides //)

Example 4

In the figure, AB and AC are the chords of a circle so that $\angle BAC$ is acute and $BA > AC$. The bisector of $\angle BAC$ meets the circle at X. Chord BE is parallel to XA. Prove that $CE = AX$.



Let $\angle CAX = \alpha = \angle BAX$

$\angle ABE = \alpha$ (alt. ∠s, AX // EB)

$\angle ACE = \alpha$ (∠s in the same seg.)

$\therefore \angle ACE = \alpha = \angle CAX$

$\overset{\frown}{AE} = \overset{\frown}{CX}$ (eq. ∠s. eq. arcs.)

$\overset{\frown}{CAE} = \overset{\frown}{ACX}$

$\therefore CE = AX$ (eq. arcs eq. chords)

Method 2

Let $\angle CAX = \alpha = \angle BAX$

$\angle ABE = \alpha$ (alt. ∠s, AX // EB)

$\angle ACE = \alpha$ (∠s in the same seg.)

$\therefore \angle ACE = \alpha = \angle CAX$

$AC = AC$ (common side)

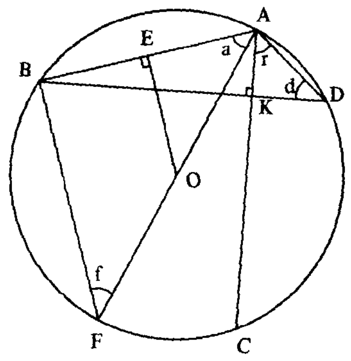
$\angle AEC = \angle AXC$ (∠s in the same segment)

$\Delta ACE \cong \Delta ACX$ (A.A.S.)

$CE = AX$ (corr. sides ≅ Δs)

Example 5

In the figure, O is the centre of the circle $ABCD$, chords $AC \perp BD$ at K, $OE \perp AB$ at E. Prove that $OE = \frac{1}{2} CD$.



$\angle ABF = 90°$	($\angle$ in semi-circle)
EO // BF	(int. $\angle$ supp)
$\Delta AEO \sim \Delta ABF$	(equiangular)
$AO = OF$	(radii)
$OE = \frac{1}{2} BF$ ······ (1)	(cor. sides, ~Δs)
$d = f$ ($\angle$s in the same segment)	
$a = 180° - 90° - f = 90° - f$	($\angle$ sum of ΔABF)
$r = 180° - 90° - d = 90° - d$	($\angle$ sum of ΔADK)
$\therefore a = r$	
$CD = BF$	(eq. $\angle$s eq. chords)
$OE = \frac{1}{2} CD$ (by (1))	