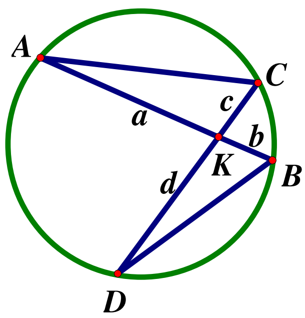
Intersecting chord theorems and other theorems on circles

Created by Francis Hung on 15 May 2010 Last updated: 22 September 2021

Theorem 1 (**Intersecting chords theorem**) AB and CD are two chords intersecting at a point K inside the circle.



If $AK = a$, $BK = b$, $CK = c$, $DK = d$, then $ab = cd$.

Proof: $\angle KAC = \angle KDB$ ($\angle$ in the same segment)

$\angle KCA = \angle KBD$ ($\angle$ in the same segment)

$\angle AKC = \angle DKB$ (vert. opp. angles)

$\Delta AKC \sim \Delta DKB$ (equiangular)

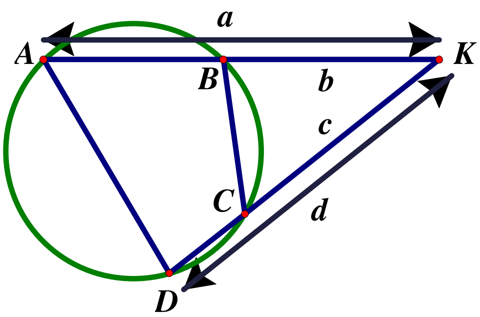
$\frac{a}{d} = \frac{c}{b}$ (corr. of sides, ~Δs)

$ab = cd$

Hence results follow.

Note that its **converse** is also true.

Theorem 2 (**Intersecting chords theorem**) AB and CD are two chords intersecting at a point K outside the circle.



If $AK = a$, $BK = b$, $CK = c$, $DK = d$, then $ab = cd$.

Proof: $\angle KAD = \angle KCB$ (ext. $\angle$, cyclic quad.)

$\angle KDA = \angle KBC$ (ext. $\angle$, cyclic quad.)

$\angle AKD = \angle CKB$ (common $\angle$)

$\Delta AKD \sim \Delta CKB$ (equiangular)

$\frac{a}{c} = \frac{d}{b}$ (corr. sides, ~Δs)

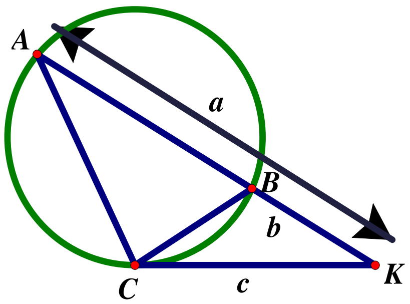
$ab = cd$

Hence results follow.

Note that its **converse** is also true.

Theorem 3 (Intersecting chords theorem)

Chord AB produced and the tangent at C intersect at a point K.



If $AK = a$, $BK = b$, $CK = c$, then $ab = c^2$.

Proof: $\angle BCK = \angle CAK$ ($\angle$ in alt. seg.)

$\angle BKC = \angle CKA$ (common $\angle$)

$\angle CBK = \angle ACK$ ($\angle$ sum of Δ)

$\Delta CKB \sim \Delta AKC$ (equiangular)

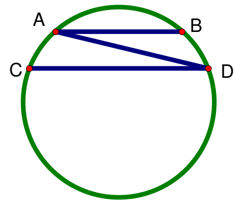
$\frac{a}{c} = \frac{c}{b}$ (corr. sides, ~Δs)

$ab = c^2$

Hence results follow.

Note that its **converse** is also true.

Theorem 4 Let $\overset{\frown}{AC}$ and $\overset{\frown}{BD}$ be two equal arcs. Then AB // CD.



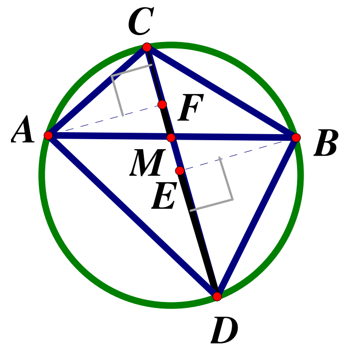
Proof: Join AD.

$\angle ADC = \angle BAD$ (eq. arcs eq. $\angle$s)

AB // CD (alt. $\angle$ eq.)

Note that its **converse** is also true.

Theorem 5 In ΔABC, M is the mid point of AB. CM meets the circumcircle ABC at D. Then $AC \cdot AD = BC \cdot BD$



Proof: Let F be the foot of perpendicular drawn from A onto CD.

Let E be the foot of perpendicular drawn from B onto CD.

$AM = MB$ (given)

$\angle AFM = \angle BEM = 90°$ (by construction)

$\angle AMF = \angle BME$ (vert. opp. $\angle$s)

$\Delta AFM \cong \Delta BEM$ (A.A.S.)

$AF = BE$ (corr. sides $\cong \Delta$s)

Area of $\Delta ACD = \frac{1}{2} AC \cdot AD \sin \angle CAD$

$= \frac{1}{2} CD \cdot AF$

$= \frac{1}{2} CD \cdot BE$

= Area of ΔBCD

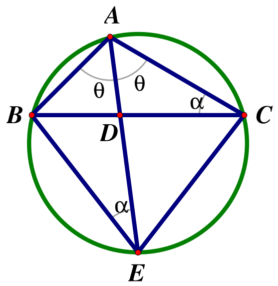
$= \frac{1}{2} BC \cdot BD \sin \angle CBD$

$\therefore \frac{1}{2} AC \cdot AD \sin \angle CAD = \frac{1}{2} BC \cdot BD \sin \angle CBD$

$\because \sin \angle CAD = \sin(180° - \angle CBD) = \sin \angle CBD$ (opp. $\angle$s, cyclic quad.)

$\therefore AC \cdot AD = BC \cdot BD$

Theorem 6 In ΔABC, the angle bisector of $\angle A$ cuts BC at D and also the circumcircle ABC at E. Prove that $AB \cdot AC = AD \cdot AE$



Proof: Let $\angle BAE = \theta = \angle CAE$ ($\angle$ bisector)

$\angle ACD = \alpha = \angle AEB$ ($\angle$s in the same seg.)

$\angle ABE = \angle ADC$ ($\angle$ sum of Δ)

$\Delta ABE \sim \Delta ADC$ (equiangular)

$\frac{AB}{AE} = \frac{AD}{AC}$ (corr. sides, ~Δs)

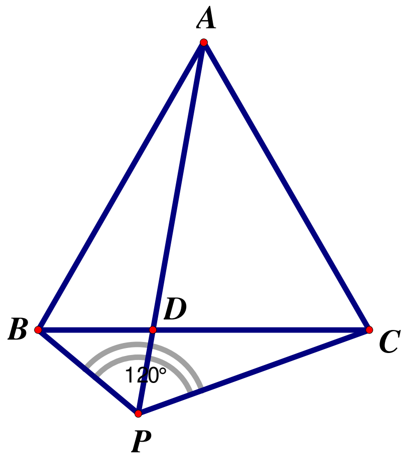
$\Rightarrow AB \cdot AC = AD \cdot AE$

Theorem 7 Given an equilateral ΔABC. P is a point outside ΔABC such that $\angle BPC = 120°$.

Prove that (a) $AP = BP + PC$

(b) If AP intersects BC at D, then $\frac{1}{PB}+\frac{1}{PC}=\frac{1}{PD}$.

Remark: the theorem is extracted from
1972 中文中學會考 高級數學 試卷二 Q8



Proof: we have several methods in proving (a).

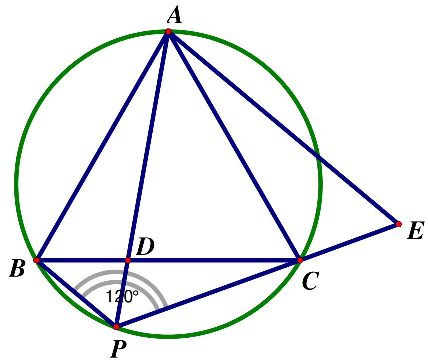
Method 1 $\because \angle BAC = 60°$, $\angle BPC = 120°$

$\angle BAC + \angle BPC = 60° + 120° = 180°$

$ABPC$ is a cyclic quad. (opp. ∠s supp.)

Construct the circumcircle $ABPC$.

Produce PC to E so that $BP = CE$



$AB = AC$ (sides of an equilateral Δ)

$\angle ABP = \angle ACE$ (ext. ∠, cyclic quad.)

$\therefore \Delta ABP \cong \Delta ACE$ (S.A.S.)

$AP = AE$ (corr. sides ≅Δs)

$\angle PAE = \angle PAC + \angle CAE$

$= \angle PAC + \angle BAP$ (corr. ∠s, ≅Δs)

$= \angle BAC = 60°$

$\therefore \Delta APE$ is an equilateral Δ.

$PE = PA$ (sides of an equilateral Δ)

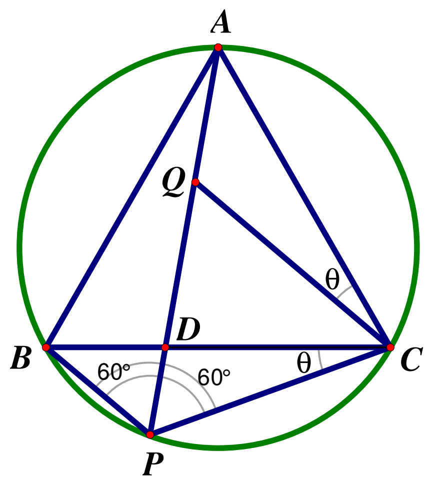
$PC + CE = PA$

$PB + PC = PA$ by construction

Method 2 As in method 1, draw the circumcircle $ABPC$.

$\angle APC = \angle APB$ (eq. chords. eq. ∠s)

$= \frac{120^\circ}{2} = 60°$



Let Q be a point on AP such that $\angle ACQ = \theta = \angle BCP$

$\angle QCP = \angle QCB + \theta = \angle ACB = 60°$

$\therefore \Delta QCP$ is an equilateral Δ.

$PC = QC$ (sides of an equilateral Δ)

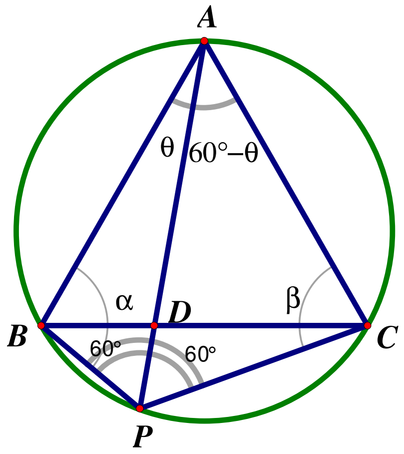
$AC = BC$ (sides of an equilateral Δ)

$\angle ACQ = \theta = \angle BCP$

$\therefore \Delta ACQ \cong \Delta BCP$ (S.A.S.)

$AQ = BP$ (corr. sides, ≅Δs)

$PA = AQ + QP = PB + CP$



Method 3 As in method 2, draw the circumcircle *ABPC*.

Let $\angle ABP = \alpha$, $\angle ACP = \beta$, $\angle BAP = \theta$, $\angle CAP = 60° - \theta$

$\alpha + \beta = 180°$ (opp. ∠s, cyclic quad.) $\Rightarrow \sin\alpha = \sin\beta \cdots (1)$

$\alpha+\theta+60° = 180°$ (∠s sum of ΔABP) $\Rightarrow \alpha = 90°+30° - \theta \cdots(2)$

Apply sine formula on ΔABP and ΔACD.

$$\frac{PA}{\sin\alpha} = \frac{PB}{\sin\theta} \Rightarrow PB = \frac{PA}{\sin\alpha}\sin\theta \quad \cdots (3)$$

$$\frac{PA}{\sin\beta} = \frac{PC}{\sin(60^\circ - \theta)} \Rightarrow PC = \frac{PA}{\sin\beta}\sin(60^\circ - \theta) \quad \cdots(4)$$

(3) + (4): $PB + PC = \frac{PA}{\sin\alpha}\sin\theta + \frac{PA}{\sin\beta}\sin(60^\circ - \theta)$

$$= \frac{PA}{\sin\alpha}\left[\sin\theta + \sin(60^\circ - \theta)\right] \text{ by (1)}$$

$$= \frac{PA}{\sin(90^\circ + 30^\circ - \theta)}\left[2\sin 30^\circ \cos(30^\circ - \theta)\right]$$

$$= \frac{PA}{\cos(30^\circ - \theta)}\left[\cos(30^\circ - \theta)\right]$$

$= PA$

To prove (b): By (a), $PB + PC = PA$

$(PB + PC)\cdot PD = PD \cdot PA \cdots\cdots(1)$

$\because$ PA is the ∠ bisector of $\angle BPC$.

By theorem 6, $PB \cdot PC = PD \cdot PA$

Sub. into (1), $(PB + PC)\cdot PD = PB \cdot PC$

$$\frac{1}{PB} + \frac{1}{PC} = \frac{1}{PD}.$$