The angle between the two diagonals of a cyclic quadrilateral

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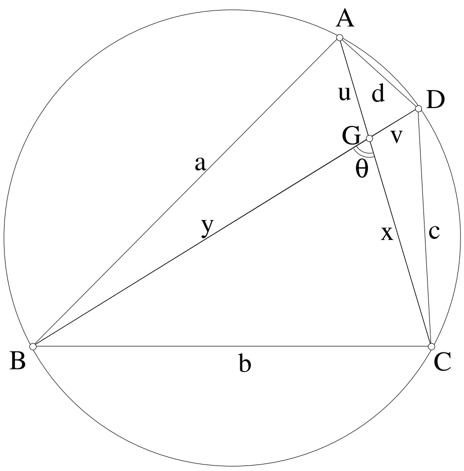

In a cyclic quadrilateral $ABCD$, $AB = a$, $BC = b$, $CD = c$, $AD = d$, AC and BD intersects at G. $AG = u$, $DG = v$, $BG = y$, $CG = x$, $\angle BGC = \theta$.

In ΔABG, $a^2 = u^2 + y^2 - 2\,uy \cos(180° - \theta)$

$\Rightarrow a^2 = u^2 + y^2 + 2\,uy \cos\theta \cdots\cdots\cdots (1)$

In ΔBCG, $b^2 = x^2 + y^2 - 2\,xy \cos\theta \cdots\cdots\cdots (2)$

(1) – (2) $\quad a^2 - b^2 = u^2 - x^2 + 2\,y\,(u + x) \cos\theta \cdots\cdots\cdots (3)$

In ΔCDG, $c^2 = x^2 + v^2 + 2\,xv \cos\theta \cdots\cdots\cdots (4)$

In ΔADG, $d^2 = u^2 + v^2 - 2\,uv \cos\theta \cdots\cdots\cdots (5)$

(4) – (5) $\quad c^2 - d^2 = x^2 - u^2 + 2\,v\,(u + x) \cos\theta \cdots\cdots\cdots (6)$

(3) + (6): $a^2 + c^2 - b^2 - d^2 = 2\,(v + y)\,(u + x) \cos\theta$

$$\therefore \cos\theta = \frac{a^2 + c^2 - b^2 - d^2}{2(u + x)(v + y)} \quad \cdots\cdots\cdots (7)$$

Recall the formula of a quadrilateral: $K = \frac{1}{2} AC \cdot BD \sin\theta$

$$\therefore \sin\theta = \frac{2K}{(u + x)(v + y)} \quad \cdots\cdots\cdots (8)$$

$$\frac{(8)}{(7)}: \quad \tan\theta = \frac{4K}{a^2 + c^2 - b^2 - d^2} = \frac{4\sqrt{(s-a)(s-b)(s-c)(s-d)}}{a^2 + c^2 - b^2 - d^2},$$

where $s = \frac{1}{2}(a + b + c + d)$, half of the perimeter.

As an exercise, if $AB = 5$, $BC = 8$, $CD = 3$, $AD = 3$, $\angle DAB = 120°$.

Prove that $ABCD$ is a cyclic quadrilateral and that the angle between the two diagonals is 60°.