Area of a cyclic quadrilateral

Created by Francis Hung on 20080225 Last updated: 02 September 2021

Reference: Additional Pure Mathematics A Modern Course Fourth Edition Volume 1 (1994)

by W. K. Chow, P.F. So, K.Y. Tam, W.K. Mui: p.166 Q19

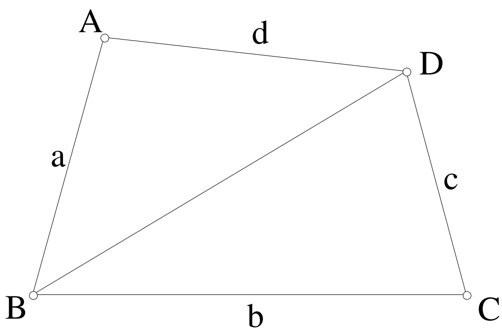
ABCD is a quadrilateral where $AB = a$, $BC = b$, $CD = c$ and $DA = d$.

(a) Express the area K of *ABCD* in terms of a, b, c, d and the angles A and C.

(b) Using the cosine law, express the length of BD in two ways in terms of a, b, c, d and the angles A and C.

(c) Show that $16K^2 + (a^2 + d^2 - b^2 - c^2)^2 = 4(a^2d^2 + b^2c^2) - 8abcd\cos(A + C)$.

(d) If the four sides of a quadrilateral are fixed in length but the shape of the quadrilateral varies, show that the area is a maximum when it is cyclic.
Hence find the maximum area in terms of a, b, c, d.



(a) K = area of *ABCD*

$= \text{area of } \Delta ABD + \text{area of } \Delta CBD$

$= \frac{1}{2}ad\sin A + \frac{1}{2}bc\sin C$

(b) By cosine law, $BD = \sqrt{a^2 + d^2 - 2ad\cos A}$

$BD = \sqrt{b^2 + c^2 - 2bc\cos C}$

(c) By (a), $2K = ad\sin A + bc\sin C$

$(2K)^2 = (ad\sin A + bc\sin C)^2$

$4K^2 = a^2d^2\sin^2 A + b^2c^2\sin^2 C + 2abcd\sin A\sin C$ ……… (1)

By (b), $BD^2 = a^2 + d^2 - 2ad\cos A = b^2 + c^2 - 2bc\cos C$

$a^2 + d^2 - b^2 - c^2 = 2(ad\cos A - bc\cos C)$

$(a^2 + d^2 - b^2 - c^2)^2 = 4(ad\cos A - bc\cos C)^2$ ……… (2)

4(1) + (2) : $16K^2 + (a^2 + d^2 - b^2 - c^2)^2$

$= 4a^2d^2\sin^2 A + 4b^2c^2\sin^2 C + 8abcd\sin A\sin C + 4(ad\cos A - bc\cos C)^2$

$= 4a^2d^2(\sin^2 A + \cos^2 A) + 4b^2c^2(\sin^2 C + \cos^2 C) + 8abcd\sin A\sin C - 8abcd\cos A\cos C$

$= 4a^2d^2 + 4b^2c^2 - 8abcd(\cos A\cos C - \sin A\sin C)$

$= 4(a^2d^2 + b^2c^2) - 8abcd\cos(A + C)$

(d) $-8abcd\cos(A+C)\le 8abcd$, equality holds when $A+C=180^\circ$

$16K^2+\left(a^2+d^2-b^2-c^2\right)^2\le 4\left(a^2d^2+b^2c^2\right)+8abcd$, equality holds when $ABCD$ is a cyclic quadrilateral.

Maximum area = $K^2=\frac{1}{16}\left[4\left(a^2d^2+b^2c^2\right)+8abcd-\left(a^2+d^2-b^2-c^2\right)^2\right]$

$$=\frac{1}{16}\left[\left(4a^2d^2+4b^2c^2+8abcd\right)-\left(a^2+d^2-b^2-c^2\right)^2\right]$$

$$=\frac{1}{16}\left[(2ad+2bc)^2-\left(a^2+d^2-b^2-c^2\right)^2\right]$$

$$=\frac{1}{16}\left(2ad+2bc+a^2+d^2-b^2-c^2\right)\cdot\left(2ad+2bc-a^2-d^2+b^2+c^2\right)$$

$$=\frac{1}{16}\left(a^2+2ad+d^2-b^2+2bc-c^2\right)\cdot\left(b^2+2bc+c^2-a^2+2ad-d^2\right)$$

$$=\frac{1}{16}\left[(a+d)^2-(b-c)^2\right]\cdot\left[(b+c)^2-(a-d)^2\right]$$

$$=\frac{1}{16}(a+d+b-c)(a+d+c-b)(a+b+c-d)(b+c+d-a)$$

Let $s=\frac{1}{2}(a+b+c+d)$, half of the perimeter. Then

$2s-2a=b+c+d-a, 2s-2b=a+c+d-b, 2s-2c=a+b+d-c, 2s-2d=a+b+c-d$

$$K^2=\frac{1}{16}(2s-2a)(2s-2b)(2s-2c)(2s-2d)$$

$$K^2=(s-a)(s-b)(s-c)(s-d)$$

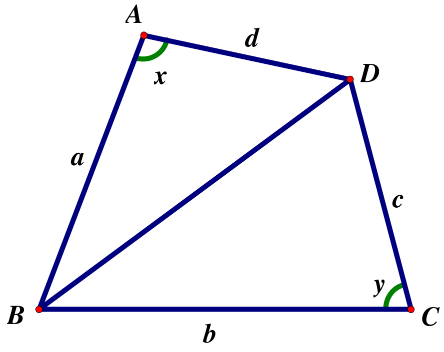
$$K=\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Note:

When d = 0, the cyclic quadrilateral *ABCD* becomes a triangle *ABC*, the area reduces to Heron's formula.

Method 2

(a) In the figure, $ABCD$ is a convex quadrilateral, $\angle A = x$, $\angle C = y$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ad\sin x}{bc\sin y}$



(b) Using area formula, show that when the area of quadrilateral $ABCD$ is a maximum, then $ABCD$ is a cyclic quadrilateral.

(c) Show that when A, B, C, D are concyclic,
$\cos x = \frac{a^2+d^2-b^2-c^2}{2(bc+ad)}$, $S_{ABCD} = \frac{1}{2}(ad+bc)\sin x$

(d) Using (c) to show that the area of a cyclic quadrilateral is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$, where $s = \frac{1}{2}(a+b+c+d)$.

(a) $BD^2 = a^2 + d^2 - 2ad\cos x = b^2 + c^2 - 2bc\cos y$

Differentiate both sides w.r.t. x

$$2ad\sin x = 2bc\sin y\frac{dy}{dx}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ad\sin x}{bc\sin y}$$

(b) Let S be the area of the quadrilateral $ABCD$.

$$S = \text{area of } \Delta ABD + \text{area of } \Delta CBD = \frac{1}{2}ad\sin x + \frac{1}{2}bc\sin y$$

$$\frac{\mathrm{d}S}{\mathrm{d}x} = \frac{1}{2}ad\cos x + \frac{1}{2}bc\cos y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}ad\cos x + \frac{1}{2}bc\cos y\cdot\frac{ad\sin x}{bc\sin y} = \frac{ad}{2}\cdot\frac{\sin y\cos x+\cos y\sin x}{\sin y}$$

$$\frac{\mathrm{d}S}{\mathrm{d}x} = \frac{ad}{2}\cdot\frac{\sin(x+y)}{\sin y} = 0;\ x + y = \pi$$

$$\frac{\mathrm{d}^2S}{\mathrm{d}x^2} = \frac{ad}{2}\cdot\frac{\sin y\cos(x+y)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)-\sin(x+y)\cos y\frac{\mathrm{d}y}{\mathrm{d}x}}{\sin^2 y}$$

$$= \frac{ad}{2}\cdot\frac{\sin y\cos(x+y)\left(1+\frac{ad\sin x}{bc\sin y}\right)-\sin(x+y)\cos y\cdot\frac{ad\sin x}{bc\sin y}}{\sin^2 y}$$

$$= \frac{ad}{2}\cdot\frac{\cos(x+y)(bc\sin y+ad\sin x)-ad\sin(x+y)\cos y\sin x}{bc\sin^3 y}$$

$$\left.\frac{\mathrm{d}^2S}{\mathrm{d}x^2}\right|_{x+y=\pi} = -\frac{ad(bc\sin y+ad\sin x)}{2bc\sin^3 y} < 0$$

$\therefore$ When the area of quadrilateral $ABCD$ is a maximum, then $ABCD$ is a cyclic quadrilateral.

(c) When $ABCD$ is a cyclic quadrilateral, $x + y = \pi$ (opp. $\angle$, cyclic quad.)

$BD^2 = a^2 + d^2 - 2ad\cos x = b^2 + c^2 - 2bc\cos y$ (cosine law)

$a^2 + d^2 - 2ad\cos x = b^2 + c^2 - 2bc\cos(\pi - x) = b^2 + c^2 + 2bc\cos x$

$$\cos x = \frac{a^2+d^2-b^2-c^2}{2(bc+ad)}$$

$$S = \text{area of } \Delta ABD + \text{area of } \Delta CBD = \frac{1}{2}ad\sin x + \frac{1}{2}bc\sin y = \frac{1}{2}(ad+bc)\sin x$$

(d) $S=\frac{1}{2}(ad+bc)\sin x=\frac{1}{2}(ad+bc)\sqrt{1-\cos^2 x}=\frac{1}{2}(ad+bc)\sqrt{(1-\cos x)(1+\cos x)}$

$S=\frac{1}{2}(ad+bc)\sqrt{\left(1-\frac{a^2+d^2-b^2-c^2}{2(bc+ad)}\right)\left(1+\frac{a^2+d^2-b^2-c^2}{2(bc+ad)}\right)}$, by ($c$)

$S=\frac{1}{4}\sqrt{\left[2(bc+ad)-\left(a^2+d^2-b^2-c^2\right)\right]\left[2(bc+ad)+\left(a^2+d^2-b^2-c^2\right)\right]}$

$S=\frac{1}{4}\sqrt{\left[(b+c)^2-(a-d)^2\right]\left[(a+d)^2-(b-c)^2\right]}$

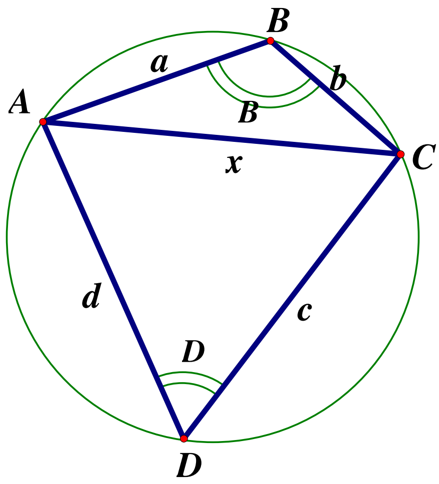
$S=\frac{1}{4}\sqrt{(a+b+c-d)(b+c+d-a)(a+b+d-c)(a+c+d-b)}$

$S=\sqrt{\frac{(a+b+c+d-2d)}{2}\frac{(a+b+c+d-2a)}{2}\frac{(a+b+c+d-2c)}{2}\frac{(a+b+c+d-2b)}{2}}$

$S=\sqrt{\frac{2S-2d}{2}\cdot\frac{2S-2a}{2}\cdot\frac{2S-2c}{2}\cdot\frac{2S-2b}{2}}$

$S=\sqrt{(s-a)(s-b)(s-c)(s-d)}$

Method 3



Join AC. Let $AB = a$, $BC = b$, $CD = c$ and $DA = d$, $AC = x$.

$\angle B + \angle D = 180°$ (opp. $\angle$s, cyclic quad.)

In ΔABC, $a^2 + b^2 - 2ab \cos B = x^2$ …… (1)

In ΔADC, $c^2 + d^2 - 2cd \cos D = x^2$ …… (2)

$\because \cos D = -\cos B$

(1) = (2): $a^2 + b^2 - 2ab \cos B = c^2 + d^2 + 2cd \cos B$

$\cos B = \dfrac{a^2+b^2-c^2-d^2}{2(ab+cd)}$

$$1-\cos^2 B = (1+\cos B)(1-\cos B)$$

$$=\left(1+\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}\right)\left(1-\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}\right)$$

$$=\left[\frac{a^2+2ab+b^2-(c^2-2cd+d^2)}{2(ab+cd)}\right]\left[\frac{c^2+2cd+d^2-(a^2-2ab+b^2)}{2(ab+cd)}\right]$$

$$=\left[\frac{(a+b)^2-(c-d)^2}{2(ab+cd)}\right]\left[\frac{(c+d)^2-(a-b)^2}{2(ab+cd)}\right]$$

$$=\left[\frac{(a+b+c-d)(a+b-c+d)}{2(ab+cd)}\right]\left[\frac{(c+d+a-b)(c+d-a+b)}{2(ab+cd)}\right]$$

$$=\frac{(2s-2a)(2s-2b)(2s-2c)(2s-2d)}{4(ab+cd)^2}\text{, where } 2s = a+b+c+d$$

$$\sin^2 B = \sin^2 D = \frac{4(s-a)(s-b)(s-c)(s-d)}{(ab+cd)^2} \Rightarrow \sin B = \sin D = \frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ab+cd}$$

$$\text{Area of } ABCD = \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin D$$

$$=\frac{1}{2}ab\cdot\frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ab+cd}+\frac{1}{2}cd\cdot\frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ab+cd}$$

$$=(ab+cd)\cdot\frac{\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ab+cd}$$

$$=\sqrt{(s-a)(s-b)(s-c)(s-d)}$$