Area of rectangle inside a circle

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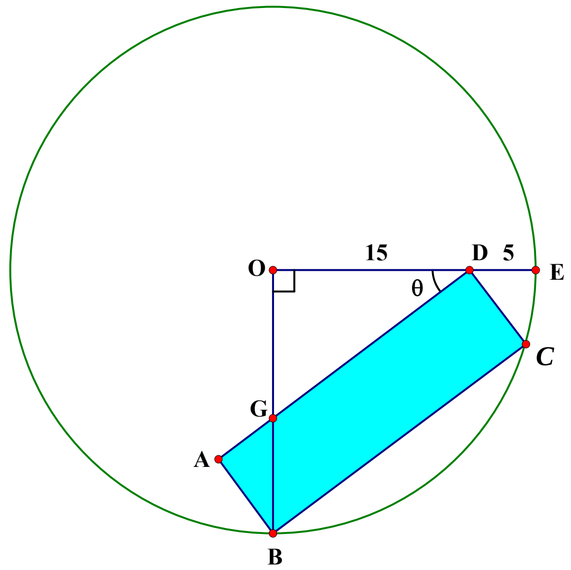
In the figure, O is the centre of a circle. D is a point on the radius OE such that $OD = 15$, $DE = 5$. $ABCD$ is a rectangle with B and C lying on the figure.

Find the area od the rectangle.



Suppose OB intersects AD at G. Join BD.

Extend CD to cut the circle again at F. Join OF.

Let $\angle ODA = \theta$

$\angle OGD = 90° - \theta$ (∠ sum of Δ)

$\angle AGB = 90° - \theta$ (vert. opp. ∠s)

$\angle BAD = 90° = \angle BCD$ (property of rectangle)

$\angle ABG = \theta$ (∠ sum of Δ)

BF is the diameter (converse, ∠ in semi-circle)

$OE = 15 + 5 = 20 = OB = OF$ (radii)

$BD^2 = OB^2 + OD^2$ (Pythagoras' theorem)

$BD = 25$

∴ B, O, F are collinear

$AB \,//\, DC$ (Property of rectangle)

$\angle DFO = \angle ABG = \theta$ (alt. ∠s, $AB \,//\, DC$)

$DF^2 = OD^2 + OF^2$ (Pythagoras' theorem)

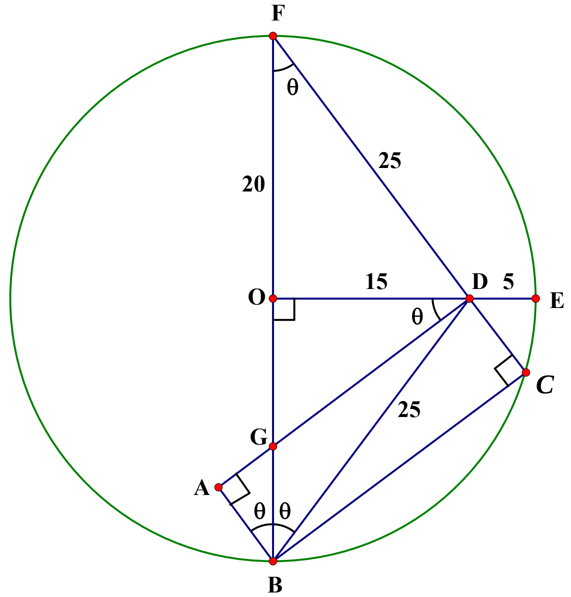
$DF = 25 = BD$

∴ ΔBDF is isosceles

$\angle DBO = \angle DFO = \theta$ (base ∠s isos. Δ)

In ΔODF, $\sin\theta = \frac{15}{25} = \frac{3}{5}$, $\cos\theta = \frac{20}{25} = \frac{4}{5}$, $\tan\theta = \frac{15}{20} = \frac{3}{4}$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2} = \frac{24}{7},\ \sin 2\theta = \frac{24}{25},\ \cos 2\theta = \frac{7}{25}$$



In ΔABD, $AB = 25\cos 2\theta = 7$, $AD = 25\sin 2\theta = 24$

Area of $ABCD = 24\times 7 = 168$

Method 2

Extend CD to cut the circle again at F. Join OF.

Extend EO to cut the circle again at H.

$\angle BCD = 90°$ (property of rectangle)

BF is the diameter (converse, ∠ in semi-circle)

$OE = 15 + 5 = 20 = OB = OF = OH$ (radii)

$BD^2 = OB^2 + OD^2$ (Pythagoras' theorem)

$BD = 25$

$DF^2 = OD^2 + OF^2$ (Pythagoras' theorem)

$DF = 25$

$CD \times DF = DE \times DH$ (intersecting chords theorem)

$CD \times 25 = 5 \times (15 + 20)$

$CD = 7$

$BC^2 + CD^2 = BD^2$ (Pythagoras' theorem)

$BC = 24$

Area of $ABCD = 24\times 7 = 168$