The radius of circumcircle of a cyclic quadrilateral

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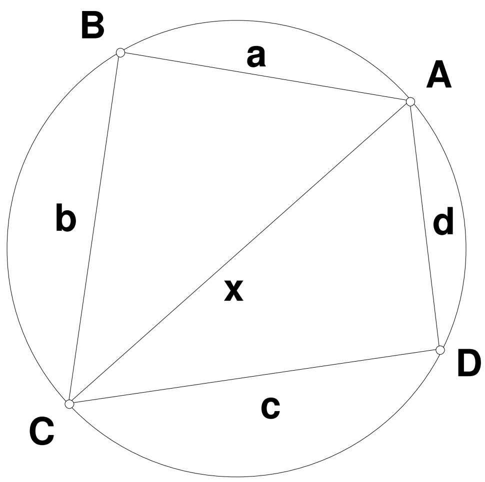

In a circle, there is a cyclic quadrilateral $ABCD$.

Let $AB = a$, $BC = b$, $CD = c$, $AD = d$, $AC = x$.

In ΔABC, $x^2 = a^2 + b^2 - 2\,ab\cos B$

In ΔADC, $x^2 = c^2 + d^2 - 2\,cd\cos D$

$\therefore x^2 = a^2 + b^2 - 2\,ab\cos B = c^2 + d^2 - 2\,cd\cos D$

$\because \angle B + \angle D = 180^\circ$ $\therefore \cos D = -\cos B$.

$a^2 + b^2 - 2\,ab\cos B = c^2 + d^2 + 2\,cd\cos B$

$$\cos B = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)}$$

$$x^2 = a^2 + b^2 - 2\,ab\,\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}$$

$$= \frac{a^3b + a^2cd + ab^3 + b^2cd - a^3b - ab^3 + abc^2 + abd^2}{ab+cd}$$

$$= \frac{ac(ad+bc) + bd(bc+ad)}{ab+cd}$$

$$x^2 = \frac{(ad+bc)(ac+bd)}{ab+cd} \Rightarrow x = \sqrt{\frac{(ad+bc)(ac+bd)}{ab+cd}}$$

Apply sine formula on ΔABC: $\frac{x}{\sin B} = 2R$, where R is the radius of the circumcircle.

$$\sin B = \frac{x}{2R} \quad \cdots\cdots\cdots(1)$$

From the notes on the area of cyclic quadrilateral, if K is the area, then

$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ where $s = \frac{1}{2}(a+b+c+d)$, half of the perimeter.

On the other hand, K = area of $ABCD$ = area of ΔABC + area of ΔACD

$$= \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin D = \frac{1}{2}(ab+cd)\sin B \quad (\because \sin D = \sin B)$$

$$\sin B = \frac{2K}{ab+cd} \quad \cdots\cdots\cdots(2)$$

Compare (1) and (2) and sub. the formula of x:

$$\sin B = \frac{\sqrt{\frac{(ad+bc)(ac+bd)}{ab+cd}}}{2R} = \frac{2K}{ab+cd}$$

$$R = \frac{1}{4K}\sqrt{(ad+bc)(ac+bd)(ab+cd)} = \frac{1}{4}\sqrt{\frac{(ad+bc)(ac+bd)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}}$$