Right-angled triangle & equilateral triangle

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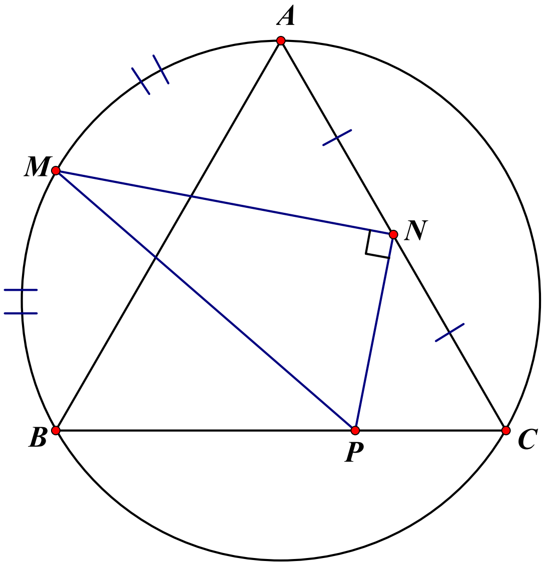

Figure A

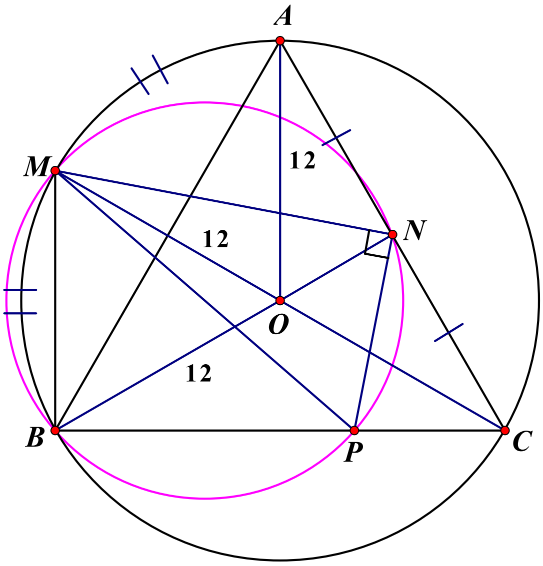

Figure B

In Figure A, an equilateral triangle ABC is inscribed in a circle of radius 12. M is the mid-point of $\overset{\frown}{AB}$ and N is the mid-point of $\overline{AC}$. P is a point on BC such that $\angle MNP = 90°$. Find the area of ΔMNP.

Solution (Figure B):

$\angle ABC = \angle ACB = \angle CAB = 60°$ (property of equilateral triangle)

Join BN and BM. Then BN is the median of the equilateral triangle ABC.

$BN \perp AC$ (property of isosceles triangle, $AB = BC$)

$\therefore$ BN is the $\perp$ bisector of AC ($\because$ N is the mid-point of AC)

$\angle ABN = \angle CBN = 30°$ (property of isosceles triangle, $AB = BC$)

Let O be the centre of the circle ABC. Join OA, OC and OM.

Then BN passes through O. i.e. B, O, N are collinear ($\perp$ bisector of chord must pass through centre)

$\angle ABM = \frac{1}{2}\angle ACB = 30°$ (M is the mid-point of $\overset{\frown}{AB}$)

$\angle MBP = \angle ABM + \angle ABC = 30° + 60° = 90°$

$\angle OBM = \angle ABM + \angle ABO = 30° + 30° = 60°$

$OB = OM = OA = 12$ (radii)

$\angle OMB = \angle OBM = 60°$ (base $\angle$s isos. Δ)

In ΔOBM, $\angle BOM = 180° - 60° - 60° = 60°$ ($\angle$ sum of Δ)

$\angle MON = 180° - \angle BOM = 120°$ (adj. $\angle$s on st. line)

$\angle BAO = \angle CAO = 30°$ ($\because$ O is the centroid of ΔABC)

In ΔAON, $\sin 30° = \frac{ON}{OA} \Rightarrow ON = 6$

In ΔMON, $MN^2 = 12^2 + 6^2 - 2\times12\times6\times\cos 120° = 252$

$MN = 6\sqrt{7}$

$\angle MNP + \angle MBP = 90° + 90° = 180°$

$\therefore$ M, B, P, N are concylcic (opp. $\angle$s supp.)

$\angle NMP = \angle NBP = 30°$ ($\angle$s in the same segment)

In ΔMNP, $\tan 30° = \frac{NP}{MN} \Rightarrow NP = \frac{6\sqrt{7}}{\sqrt{3}}$

Area of $\Delta MNP = \frac{1}{2}MN\times NP = \frac{1}{2}\times 6\sqrt{7}\times\frac{6\sqrt{7}}{\sqrt{3}} = 42\sqrt{3}$