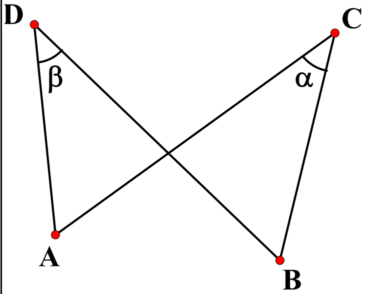
Test for Concyclic points: Converse, angles in the same segment

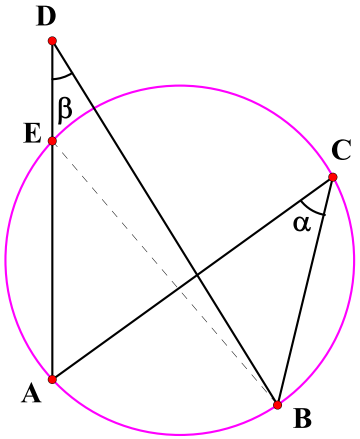
Created by Mr. Francis Hung on 20210915



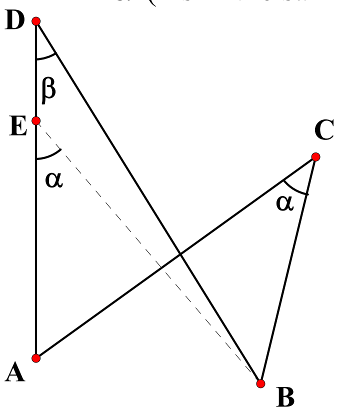
In the figure,
if α = β, then
A, *B*, *C*, *D* are concyclic.
Abbreviation:
Converse,
∠s in the same segment

Proof: Assume that any 3 points are not collinear. Draw a circle which passes through *A*, *B* and *C*. There are 3 different cases:

Case 1 *D* lies outside the circle



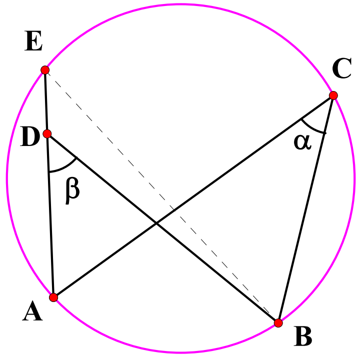
AD cuts the circle at *E* .
Join *BE* .
∠*AEB* = α (∠s in the same segment *ACB*)



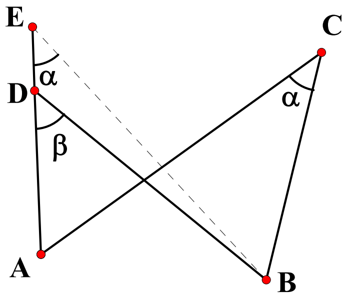
∵ α = β (given)

∴ *BE* // *BD* (corr. ∠s eq.)
This is impossible because there are two different parallel lines which meet at *B*.
∴ Case 1 is impossible.

Case 2 *D* lies inside the circle



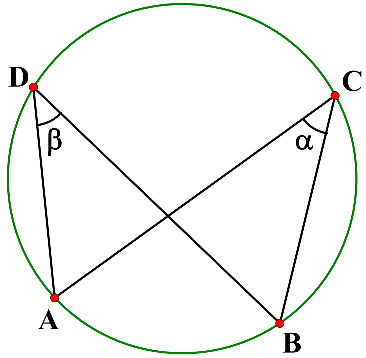
Produce *AD* which cuts the circle at *E* .
Join *BE* .
∠*AEB* = α (∠s in the same segment *ACB*)



∵ α = β (given)

∴ *BE* // *BD* (corr. ∠s eq.)
This is impossible because there are two different parallel lines which meet at *B*.
∴ Case 2 is impossible.

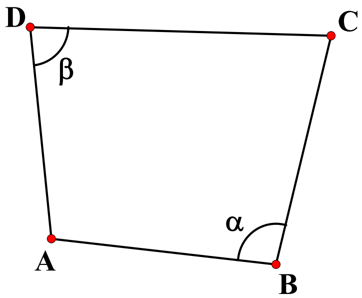
Case 3 *D* lies on the circle *ABC*.



∵ Case 1 and case 2 are impossible

∴ The only possible case is case 3.
i.e. *D* must lie on the circle
This is known as indirect proof.
(反證法)
∴ *A*, *B*, *C*, *D* are concyclic.

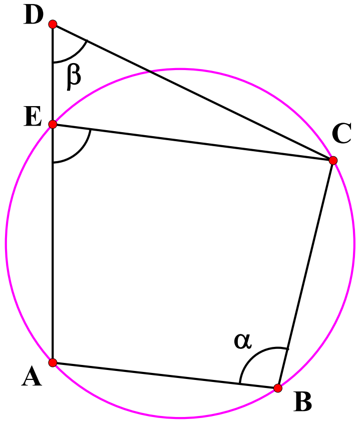
Test for Concyclic points: Opposite angles supplementary



In the figure,
if $\alpha + \beta = 180°$, then
A, *B*, *C*, *D* are concyclic.
Abbreviation: **opp. ∠s supp.**

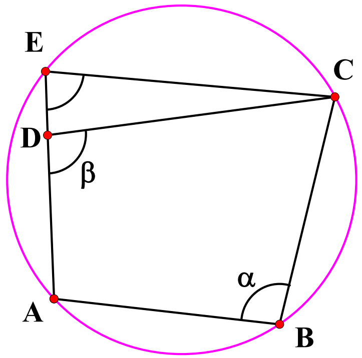
Proof: Assume that any 3 points are not collinear. Draw a circle which passes through *A*, *B* and *C*. There are 3 different cases:

Case 1 *D* lies outside the circle



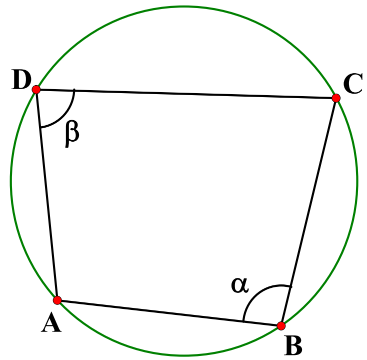
∴ Case 1 is impossible.

Case 2 *D* lies inside the circle



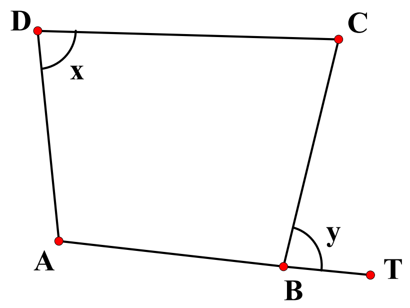
∴ Case 2 is impossible.

Case 3 *D* lies on the circle *ABC*.



∴ *A*, *B*, *C*, *D* are concyclic.

Test for Concyclic points: Exterior angle = interior opposite angle



In the figure, if $x = y$, then A, B, C, D are concyclic

Proof: $\angle ABC = 180° - y$ (adj. $\angle$s on st. line)

$\angle ABC + \angle ADC = x + 180° - y$

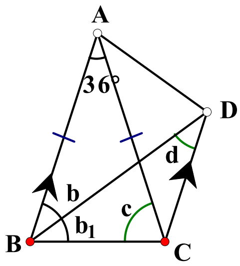
$= 180°$

$\therefore$ A, B, C, D are concyclic (opp. $\angle$s supp.)

The theorem is proved.

Abbreviation: **ext. $\angle$ = int. opp. $\angle$**

Example 1



In the figure, Given ΔABC, $AB = AC$, $\angle BAC = 36°$, the line bisecting $\angle ABC$ meets the line through C parallel to BA at D. To prove: A, B, C, D are concyclic.

Proof: $\angle ABC = \angle ACB$ (base $\angle$s isos. Δ)

$= \dfrac{180° - 36°}{2} = 72°$ ($\angle$ sum of Δ)

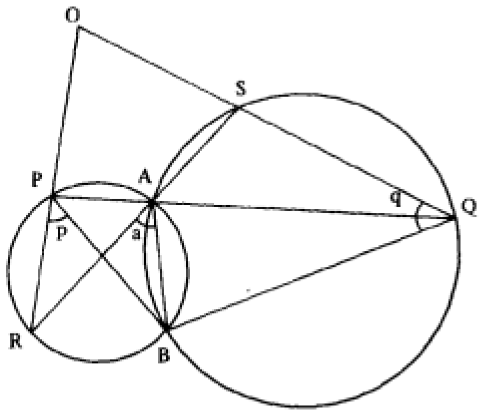
$b = b_1 = \dfrac{72°}{2} = 36°$ ($\angle$ bisector)

$d = b = 36°$ (alt. $\angle$s, AB // DC)

$\therefore$ $\angle A = \angle D$

$\therefore$ A, B, C, D are concyclic (converse, $\angle$s in the same segment)

Example 2



In the figure, 2 circles $APRB$, $ASQB$ intersect at A and B. PAQ, RAS are straight lines, RP and QS are produced to meet at O. To prove: O, P, B, Q are concyclic.

Proof: $p = a$ ($\angle$s in the same segment)

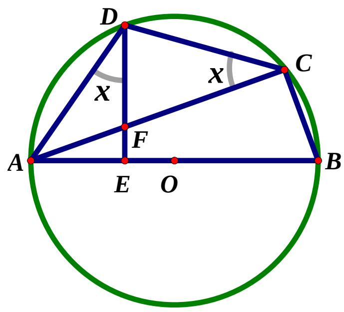
$a = q$ ($\angle$ sum of Δ)

$\therefore$ $p = q$

$\therefore$ O, P, B, Q are concyclic. (ext. $\angle$ = int. opp. $\angle$)

Example 3

In the figure, O is the centre of the circle. AOB is a diameter. AC intersects DE at F. If $\angle ADE = \angle DCA = x$, prove that $FEBC$ is a cyclic quadrilateral.



$\angle ACB = 90°$ ($\angle$ in semi-circle)

$\angle DAE = 180° - \angle BCD$ (opp. $\angle$ cyclic quad.)

$= 180° - (90° + x) = 90° - x$

$\angle AED = 180° - (90° - x) - x = 90°$ ($\angle$ sum of Δ)

$\therefore$ $\angle ACB = 90° = \angle AED$

B, C, F, E are concyclic (ext. $\angle$ = int. opp. $\angle$)

Orthocentre

Example 4

The **three altitudes** of a **triangle** are **concurrent** at a point called the "**orthocentre**". (Figure 1)

In ΔABC, $AD \perp BC$, $BE \perp AC$, $CF \perp AB$.

Then *AD*, *BE*, *CF* are **concurrent** at *H*.

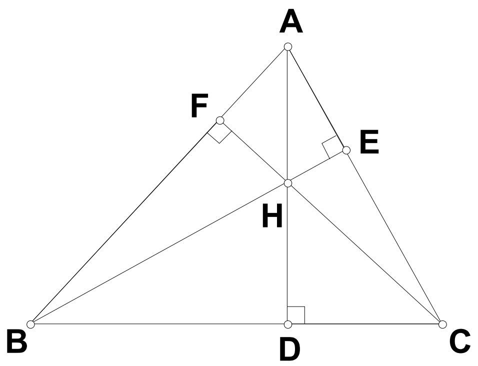


Figure 1

Proof: Let the **altitudes** *BE* and *CF* meet at *H*.

Join *AH* and produce it to meet *BC* at *D*.

Try **to show** that $AD \perp BC$. (Figure 2)

$\angle AFH + \angle AEH = 180°$

A, *F*, *H*, *E* are concyclic. (opp. ∠ supp.)

$\angle BFC = \angle BEC$

B, *C*, *E*, *F* are concyclic. (converse, ∠s in the same seg.)

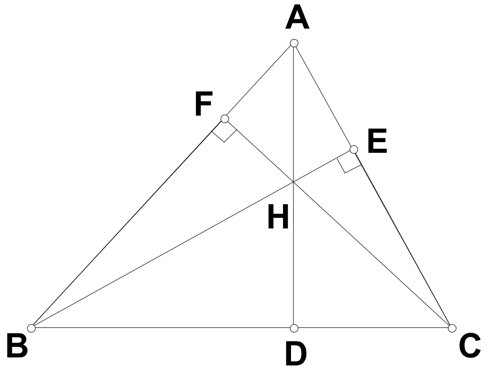


Figure 2

In **Figure 3**, let $\angle BAD = x$, $\angle AHF = y$.

$\angle FEH = x$ (∠s in the same seg.)

$\angle BCF = \angle BEF$ (∠s in the same seg.)

$= x$

$\angle CHD = y$ (vert. opp. ∠s.)

In ΔAFH, $x + y = 90°$ (∠ sum of Δ)

In ΔCDH, $x + y + \angle CDH = 180°$ (∠ sum of Δ)

$\therefore \angle CDH = 90°$

The **theorem is proved.**

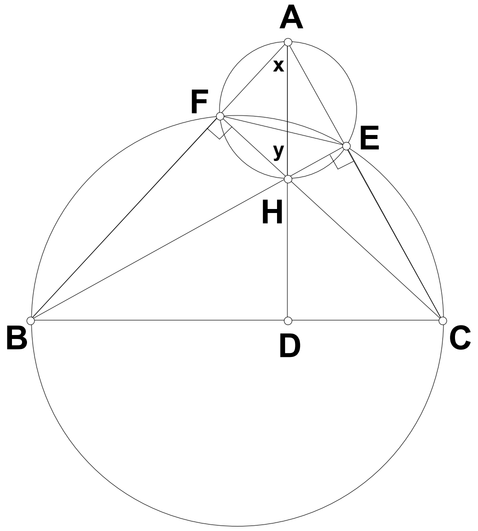


Figure 3