Concyclic points Example 2
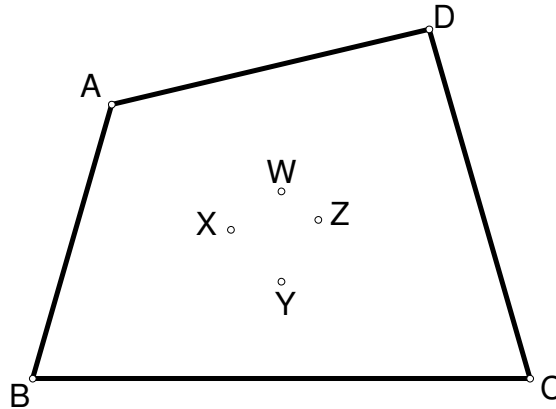
1968 Certificate of Education Examination (Chinese) General Mathematics Paper 2

Section B Q13(b)

In the figure, *ABCD* is a quadrilateral. Four circles with centres *X*, *Y*, *Z* and *W* touch 3 sides of *ABCD*. Prove that *X*, *Y*, *Z* and *W* are concyclic.

Solution: Suppose *X* is the centre of circle touching *BC*, *AB* and *DA* respectively. Then it can be easily proved that *AX*, *BX* are the angle bisectors of ∠*A* and ∠*B* respectively. Similarly, *WC* and *YD* are the angle bisectors of ∠*C* and ∠*D* respectively.
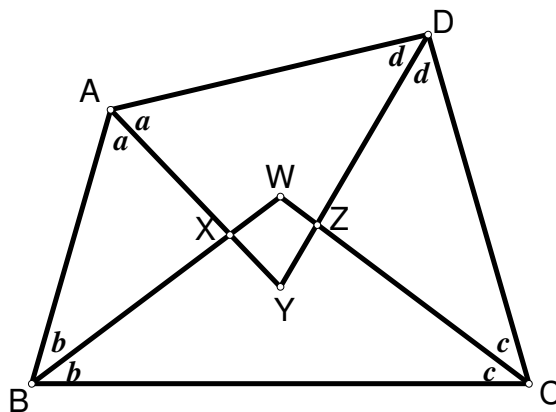
Let $\angle XAD = \angle XAB = a$, $\angle XBA = \angle XBC = b$, $\angle WCB = \angle WCD = c$, $\angle YDC = \angle YDA = d$

$\angle BWC = 180° - b - c$; $\angle AYD = 180° - a - d$ (∠ sum of Δ)

$$\begin{aligned}
\angle XWZ + \angle XYZ &= 180° - b - c + 180° - a - d \\
&= 360° - (a + b + c + d) \\
&= 360° - 180° \text{ (∠ sum of polygon)} \\
&= 180°
\end{aligned}$$

∴ *X*, *Y*, *Z* and *W* are concyclic. (opp. ∠s supp.)