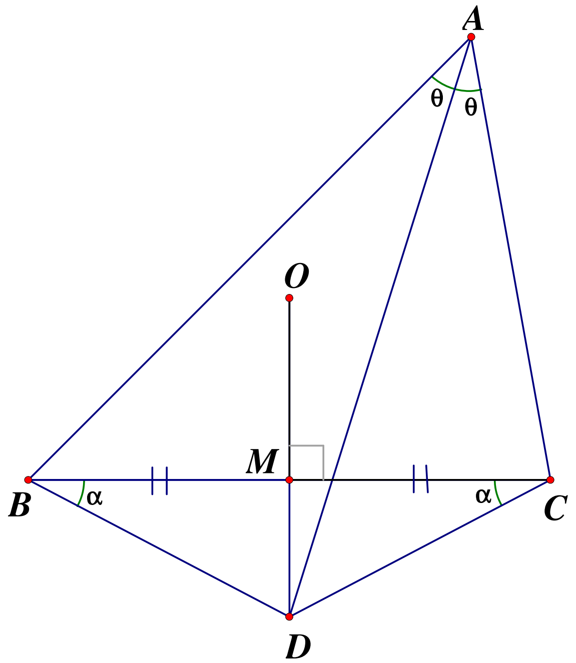
Concyclic Example 4

Created by Francis Hung Last updated: 03 September 2021

In ΔABC, the angle bisector of $\angle BAC$ and the perpendicular bisector of BC intersect at D, M is the mid-point of BC. If $AB \neq AC$, prove that $ABDC$ is a cyclic quadrilateral.



Let $\angle BAD = \theta = \angle CAD$.

$BM = MC$, $BC \perp MD$

$\angle BAC = 2\theta$

It is easy to show that $\Delta BMD \cong \Delta CMD$ (S.A.S.)

$\therefore BD = CD$ (corr. sides $\cong \Delta$s)

ΔBCD is isosceles (2 sides equal)

Let $\angle DBM = \alpha = \angle DCM$ (base $\angle$s isos. Δ)

$\frac{AD}{\sin(\alpha + B)} = \frac{BD}{\sin\theta}$ and $\frac{AD}{\sin(\alpha + C)} = \frac{CD}{\sin\theta}$ (sine rule on ΔABD and ΔACD)

$\because BD = CD$

$\therefore \frac{AD}{\sin(\alpha + B)} = \frac{BD}{\sin\theta} = \frac{AD}{\sin(\alpha + C)}$

$\sin(\alpha + B) = \sin(\alpha + C)$

$\alpha + B = \alpha + C$ or $\alpha + B + \alpha + C = 180°$

$\because B \neq C$ $\therefore \angle ABD + \angle ACD = 180°$

$ABDC$ is a cyclic quadrilateral (opp. $\angle$s supp.)

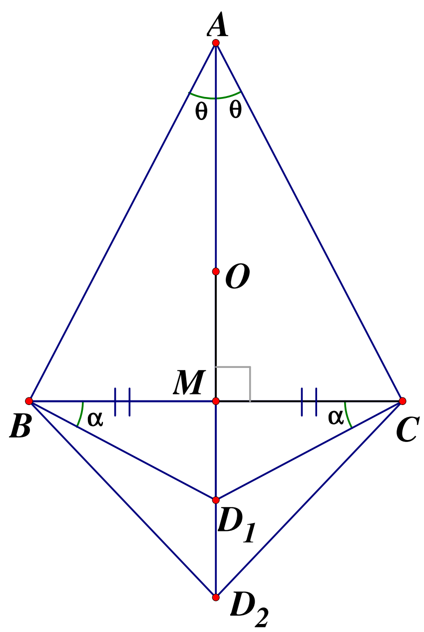
Remark:

If $AB = AC$, then the angle bisector and the perpendicular bisector will overlap.

They will intersect at infinitely many points.

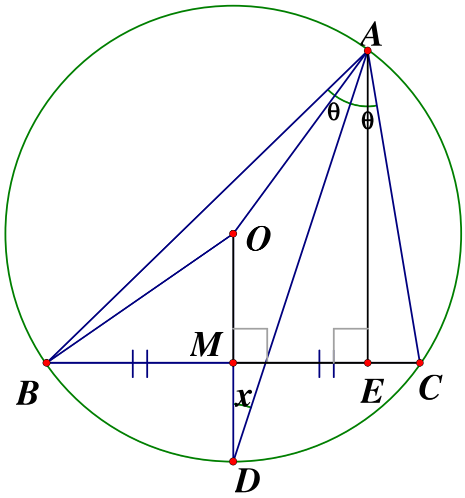
We can find infinitely many points D_1, D_2, $\cdots$.

A, B, D, C are not necessary concyclic.



Method 2

Let the circumcentre be O. Let E be the projection of A on BC. Join OA, AE, OB. Let $\angle ODA = x$.



$\angle DAE = x$ (alt. $\angle$s, OD // AE)

$\angle CAE = \theta - x$

$\angle ACE = 90° - \theta + x$ ($\angle$ sum of Δ)

$\angle AOB = 2\angle ACE$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$= 180° - 2\theta + 2x$

$OA = OB$ = radii

$\angle OAB = (180° - \angle AOB) \div 2$ ($\angle$ sum of Δ)

$= \theta - x$

$\angle OAD = \angle BAD - \angle OAB = x$

$\therefore \angle ODA = \angle OAD$

$OA = OD$ (sides opp. eq. $\angle$s)

So $OA = OB = OC = OD$

A, B, C, D are concyclic.