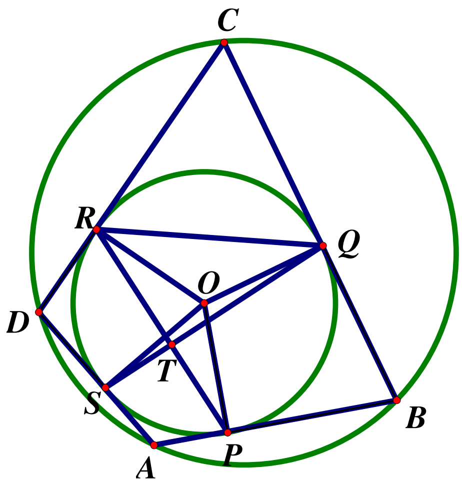
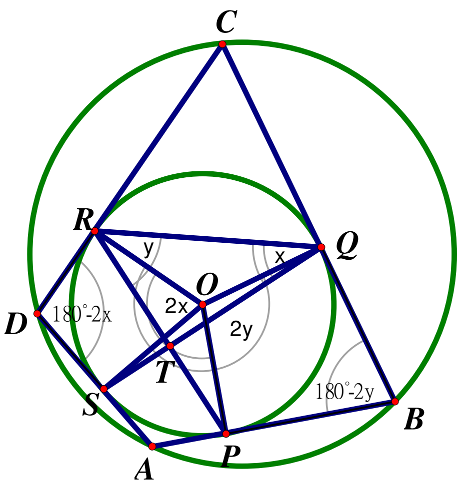
Bicentric quadrilateral

Modified from 1954 HKU O level Paper 2 Q1

Created by Francis Hung on 20080814 Last updated: 22 September 2021



In the figure, the cyclic quadrilateral *ABCD* touches the smaller inner circle with centre O at *P*, *Q*, *R* and *S*. **Prove that *SQ* ⊥ *PR*.**



Solution:

Suppose *PR* intersects *SQ* at *T*. Join *QR*.

Let $\angle TQR = x$, $\angle TRQ = y$.

$\angle SOR = 2x$, $\angle POQ = 2y$ (∠ at centre twice ∠ at ⊙^{ce}.)

$\angle ORD = \angle OSD = \angle OPB = \angle OQB = 90°$ (tangent ⊥ radius)

$\therefore ORDS$, $OQBP$ are cyclic quadrilaterals. (opp. ∠s supp.)

$\angle RDS = 180° - 2x$, $\angle QBP = 180° - 2y$ (opp. ∠s, cyclic quadrilateral)

$\because \angle RDS + \angle QBP = 180°$ (opp. ∠s, cyclic quadrilateral)

$\therefore 180° - 2x + 180° - 2y = 180°$

$x + y = 90°$

$\angle QTR = 180° - x - y$ (∠ sum of Δ)

$= 90°$

$\therefore SQ \perp PR$

If *PQ*, *RS* are parallel chords of a circle whose centre is *O*,

prove, by joining *P* to *S*, that $\angle POR = \angle SOQ$.

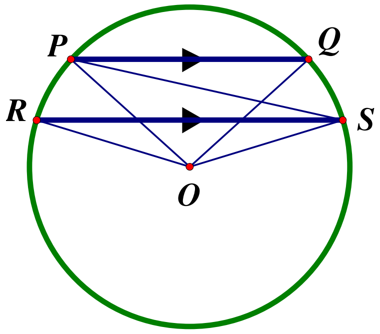
If *AB*, *CD* are perpendicular chords of the circle which intersect each other inside the circle, prove, by drawing a chord through *C* and parallel to *AB*, that $\angle AOD + \angle BOC = 180°$. Hence, or otherwise prove that the tangents to the circle at the points *A*, *D*, *B*, *C* form a cyclic quadrilateral.

Join *PS*, *PO*, *QO*, *RO*, *SO*.

$\angle QPS = \angle RSP$ (alt. ∠s *PQ* // *RS*)

$\overset{\frown}{PR} = \overset{\frown}{QS}$ (eq. ∠s eq. arcs)

$\angle POR = \angle SOQ$ (eq. arcs eq. ∠s)



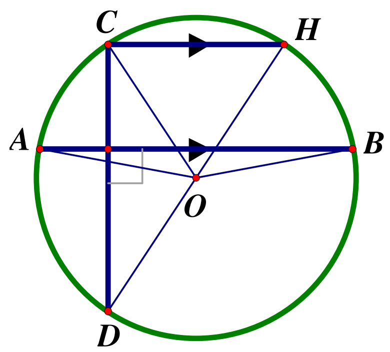
From *C*, draw a chord *CH* which is parallel to *AB*.

$\angle DCH = 90°$ (corr. ∠s *AB* // *CH*)

DH is a diameter (converse, ∠ in semi-circle)

By the above result, $\angle AOC = \angle BOH$

$\angle AOD + \angle BOC = \angle AOD + \angle AOH = 180°$



Let *O* be the centre and the quadrilateral *PQRS* touches the circle at *A*, *B*, *C*, *D*.

$OA \perp PS$, $OD \perp PQ$, $OB \perp QR$, $OC \perp RS$ (tangent ⊥ radius)

$\angle P + \angle AOD = 180°$, $\angle R + \angle BOC = 180°$ (∠ sum of polygon)

By the above result, $\angle AOD + \angle BOC = 180°$

$\therefore \angle P + \angle R = 180°$

∴ *PQRS* is a cyclic quadrilateral (opp. ∠s supp.)