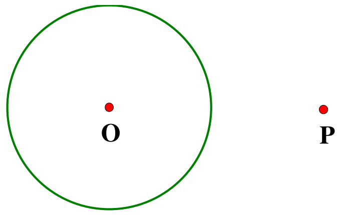
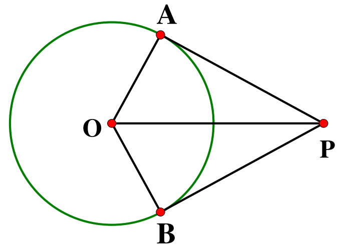
Tangent Property (Tangent from External Point)

Created by Mr. Francis Hung on 20210922 Last updated: 2021-09-26



In the figure, P is a point outside a circle with centre at O.

P is called an external point of the circle.



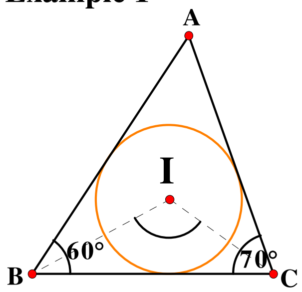
Two tangents can be drawn through P, touching the circle at A and B.

Join OA, OB and OP.

$OA = OB$	(radii)
$\angle OAP = 90°$	(tangent $\perp$ radius)
$\angle OBP = 90°$	(tangent $\perp$ radius)
$OP = OP$	(common sides)
$\therefore \Delta OAP \cong \Delta OBP$	(R.H.S.)
$\therefore$ **$\angle APO = \angle BPO$** $\cdots$(1)	(cor. $\angle$s, $\cong \Delta$s)
$\angle AOP = \angle BOP$ $\cdots$(2)	(cor. $\angle$s, $\cong \Delta$s)
$PA = BP$ $\cdots\cdots$(3)	(cor. sides, $\cong \Delta$s)

The above three results are called **tangent property** or **tangent from external point** (**tangent from ext. pt.** in short)

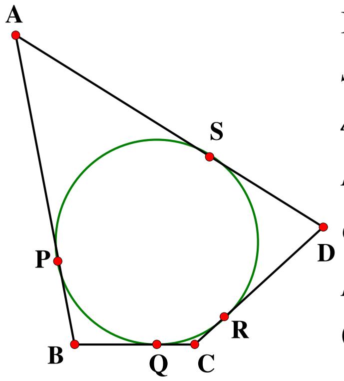
Example 1



In the figure, I is the incentre of ΔABC. $\angle B = 60°$, $\angle C = 70°$. Find $\angle BIC$.

$\angle IBA = \angle IBC = 30°$	(tangent property)
$\angle ICA = \angle ICB = 35°$	(tangent property)
In ΔIBC,	
$\angle BIC = 180° - 30° - 35°$	
$= 115°$	($\angle$ sum of Δ)

Example 2

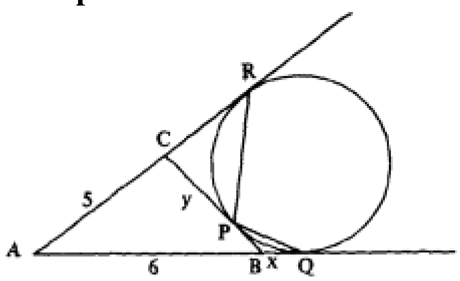


In the figure, a circle is inscribed inside a quadrilateral $ABCD$. P, Q, R and S are points of contact as shown. To prove that $AB + CD = BC + AD$.

$AP = AS$ $\cdots\cdots$(1)	(tangent property)
$BP = BQ$ $\cdots\cdots$(2)	(tangent property)
$CR = CQ$ $\cdots\cdots$(3)	(tangent property)
$DR = DS$ $\cdots\cdots$(4)	(tangent property)

(1) + (2) + (3) + (4): $AB + CD = BC + AD$

Example 3



In the figure, a circle a circle touches AB at Q, BC at P and AC at R. $AB = 6$ cm, $AC = 5$ cm, $BC = 4$ cm. To find CP and AQ.

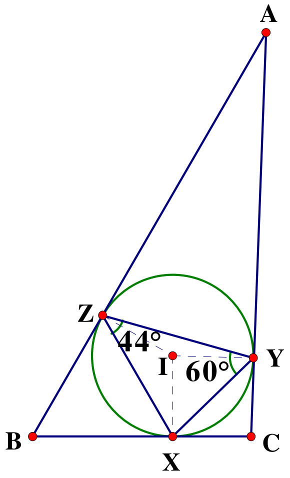
Using the notation in the figure,

$CR = y$	(tangent properties)
$BP = x = 4 - y$	(tangent properties)
$AR = AQ \Rightarrow 5 + y = 6 + x = 10 - y$	(tangent properties)

$y = 2.5$, $x = 4 - y = 1.5$

$CP = 2.5$ cm, $AQ = 6 + x = 7.5$ cm

Example 4



In the figure, AB, BC, AC are tangents to the circle, centre I, at Z, X, Y respectively, $\angle XYZ = 60°$, $\angle XZY = 44°$. To find $\angle XIZ$ and angles of ΔABC.

$\angle XIZ = 2\angle XYZ$ (∠ at centre twice ∠ at ⊙[ce])

$= 2\times 60° = 120°$

$\angle XIY = 2\angle XZY = 88°$ (∠ at centre twice ∠ at ⊙[ce])

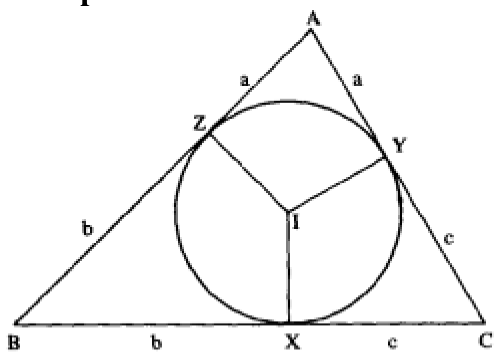
$IX \perp BC$, $IZ \perp AB$ (tangent ⊥ radius)

$\angle B = 360° - 90° - 90° - 120° = 60°$ (∠ sum of polygon)

$\angle C = 360° - 90° - 90° - 88° = 92°$ (∠ sum of polygon)

$\angle A = 180° - \angle B - \angle C = 28°$ (∠ sum of Δ)

Example 5



In the figure, AB, BC, AC are tangents to the circle, centre I, at Z, X, Y respectively, $AB = 10$ cm, $BC = 9$ cm, $AC = 7$ cm. To find BZ.

With the notation in the figure,

$AY = AZ = a$, $BX = BZ = b$, $CX = CY = c$ (tangent properties)

$a + b = 10$ ⋯⋯ (1)

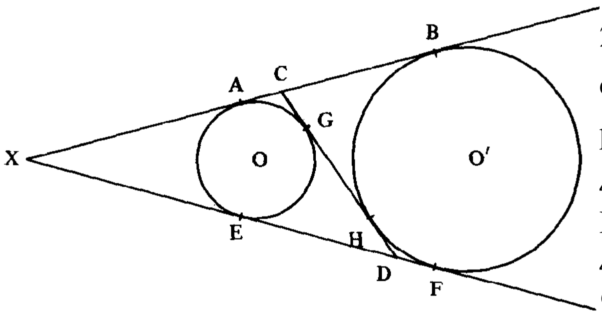
$b + c = 9$ ⋯⋯ (2)

$c + a = 7$ ⋯⋯ (3)

(1) + (2) – (3): $2b = 12$

$b = 6 \Rightarrow BZ = 6$ cm

Example 6



In the figure, AB and EF are the common tangents to 2 circles, centre O and O' . GH is the transverse common tangent to the two circles at G and H. GH id produced to cut EF at D and HG is produced to cut AB at C. To prove $AB = CD$.

Produce BA to meet FE at X.

$AC = CG$ (tangent properties)

$CH = CB$ (tangent properties)

$HD = DF$ (tangent properties)

$GD = DE$ (tangent properties)

$XB = XF$ (tangent properties)

$XA = XE$ (tangent properties)

$AB = XB - XA$

$= XF - XE$

$= EF$

$AB + EF = AC + CB + ED + DF$

$= CG + CH + GD + HD$

$= (CG + GD) + (CH + HD)$

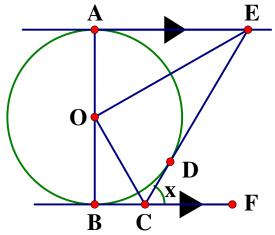
$= CD + CD = 2CD$

$\therefore AB = CD$

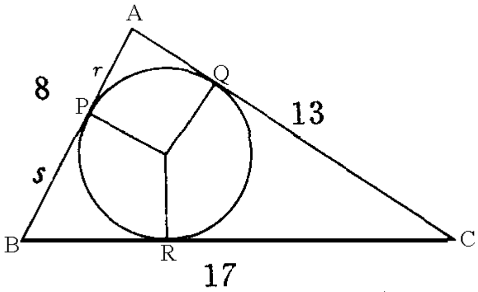
Class work on tangent properties

Created by Mr. Francis Hung on 20210904 Last updated: 2021-09-26

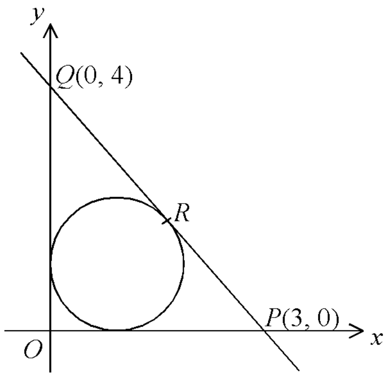
1. In the figure, *AE* and *BF* are two parallel tangents to the circle at *A* and *B* respectively. *EC* is another tangent to the circle at *D*, where *C* lies on *BF*. Suppose $\angle ECF = x$.
   (a) Express $\angle OCE$ and $\angle OEC$ in terms of *x*.
   (b) Hence, find $\angle EOC$.



2. In the figure, a circle is inscribed in ΔABC with $AB = 8$, $BC = 17$, $CA = 13$. The point of contact *P* divides *AB* in the ratio $r : s$, where $r < s$. Find $r : s$.



3. In the figure, a circle is inscribed inside the triangle *OPQ*, where *P*(3, 0), *Q*(0, 4).

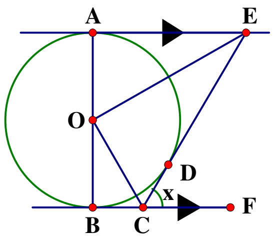


   (a) Find the radius of the inscribed circle.
   (b) If *PQ* touches the circle at *R*, find *R*.

Solution to Class work on tangent properties

Created by Mr. Francis Hung on 20210904 Last updated: 2021-09-26

1. In the figure, AE and BF are two parallel tangents to the circle at A and B respectively. EC is another tangent to the circle at D, where C lies on BF. Suppose $\angle ECF = x$.



(a) Express $\angle OCE$ and $\angle OEC$ in terms of x.

(b) Hence, find $\angle EOC$.

(a) $\angle OCE = \angle OCB$ (tangent properties)

$\angle OCE + \angle OCB + x = 180°$ (adj. ∠s on st. line)

$\angle OCE = 90° - \dfrac{x}{2}$

$\angle AEC = \angle ECF = x$ (alt. ∠s, AE // BF)

$\angle AEO = \angle OEC$ (tangent properties)

$\angle OEC = \dfrac{x}{2}$

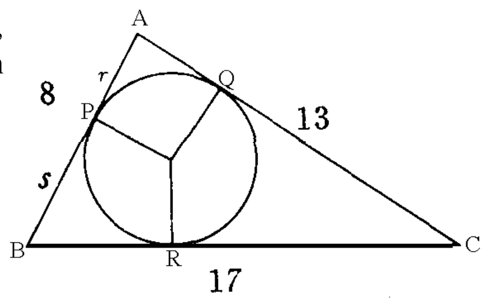
(b) In ΔOCE,

$\angle OCE + \angle CEO + \angle EOC = 180°$ (∠ sum of Δ)

$90° - \dfrac{x}{2} + \dfrac{x}{2} + \angle EOC = 180°$

$\angle EOC = 90°$

2. In the figure, a circle is inscribed in ΔABC with $AB = 8$, $BC = 17$, $CA = 13$. The point of contact P divides AB in the ratio $r : s$, where $r < s$. Find $r : s$.



Let $AP = x$. Then

$BP = 8 - x$

$AQ = x$ (tangent properties)

$CQ = 13 - x$

$BR = BP = 8 - x$ (tangent properties)

$CR = CQ = 13 - x$ (tangent properties)

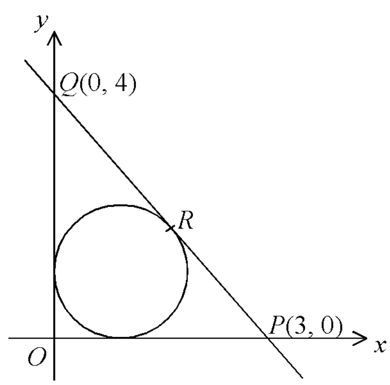
$BR + CR = BC$

$8 - x + 13 - x = 17$

$x = 2$, $BP = 8 - x = 8 - 2 = 6$

$r : s = 2 : 6 = 1 : 3$

3. In the figure, a circle is inscribed inside the triangle OPQ, where $P(3, 0)$, $Q(0, 4)$.



(a) Find the radius of the inscribed circle.

(b) If PQ touches the circle at R, find R.

$PQ = \sqrt{3^2 + 4^2} = 5 \Rightarrow PQ = 5$ (Pythagoras'theorem)

Assume the circle (radius r) touches x-axis at A and y-axis at B.

$OA = OB = r$

$AP = 3 - r = PR$ (tangent properties)

$QB = 4 - r = QR$ (tangent properties)

$PR + QR = PQ$

$3 - r + 4 - r = 5$

$r = 1$

$PR = 3 - r = 2$

$QR = 4 - r = 3$

$QR : RP = 3 : 2$

By the point of division formula,

$R = \left(\dfrac{0\times 2 + 3\times 3}{5}, \dfrac{4\times 2 + 3\times 0}{5}\right) = \left(\dfrac{9}{5}, \dfrac{8}{5}\right)$.