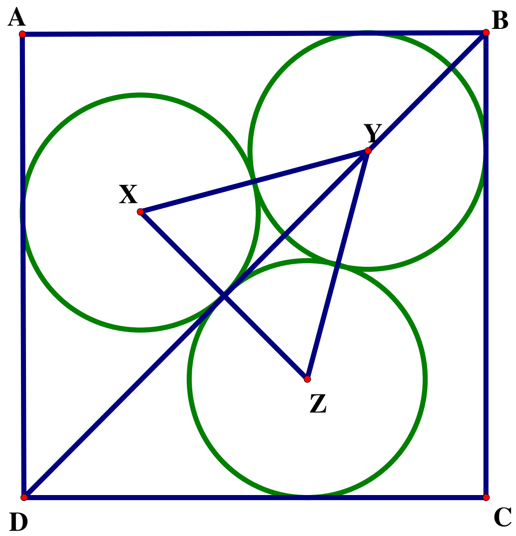
Tangent Problem

HKCEE 1973 syllabus B Paper 2 Q17

Created by Francis Hung on 20070819

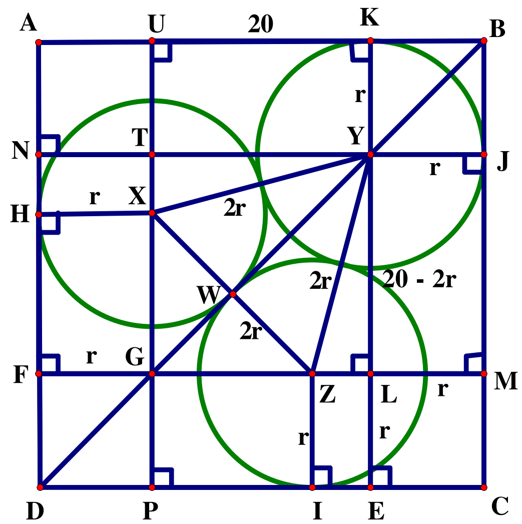
The figure shows 3 equal circles, with centres X, Y and Z touching one another and the sides of the square $ABCD$ such that the whole figure is symmetrical about the diagonal BD.



(a) What is the measure of the angle between the line AB and the line XY?

(b) If $AB = 20$, what is the length of the radius of the circle?

(a) Let the radii be r.
Join XYZ, then $XY = YZ = YZ = 2r$
$\angle XYZ = \angle YXZ = \angle XZY = 60°$
Join the centres X, Y, Z to the points of contacts I, H, K, J as shown.
Then $ZI \perp CD$, $XH \perp AD$, $YK \perp AB$, $YJ \perp BC$ (tangent $\perp$ radius)
$ZI = XH = YK = YJ = r$ (radii)
Draw $NJ \mathbin{//} FM \mathbin{//} AB \mathbin{//} DC$
Draw $PU \mathbin{//} KE \mathbin{//} AD \mathbin{//} BC$
Let T, Y, L, G be the intersections as shown.
$FG = r = LM = LE = YJ = NT = TU$
$YL = 20 - 2r = TY = TG = GL$
$YTGL$ is a square with sides $= 20 - 2r$
$\angle GYL = 45° = \angle GYT$ (property of square)
Suppose YG intersect XZ at W.
$\Delta XYW \cong \Delta ZYW$ (S.S.S.)
$\angle XYW = 30° = \angle ZYW$ (corr. $\angle$s. $\cong \Delta$s)
$\angle XYT = 45° - 30° = 15°$
$\therefore$ AB makes $15°$ with XY.



(b) In ΔTYX, $\cos 15° = \dfrac{20-2r}{2r} = \dfrac{10}{r} - 1$

$$1 + \cos 15° = \frac{10}{r}$$

$$r = \frac{10}{1+\cos 15^\circ}$$

$$= \frac{10}{1+\frac{\sqrt{6}+\sqrt{2}}{4}} = \frac{40}{4+\sqrt{6}+\sqrt{2}}$$

$$= \frac{40}{4+\sqrt{6}+\sqrt{2}} \cdot \frac{4+\sqrt{6}-\sqrt{2}}{4+\sqrt{6}-\sqrt{2}}$$

$$= \frac{40\left(4+\sqrt{6}-\sqrt{2}\right)}{\left(4+\sqrt{6}\right)^2 - 2}$$

$$= \frac{40\left(4+\sqrt{6}-\sqrt{2}\right)}{20+8\sqrt{6}} = \frac{10\left(4+\sqrt{6}-\sqrt{2}\right)}{5+2\sqrt{6}}$$

$$= \frac{10\left(4+\sqrt{6}-\sqrt{2}\right)}{5+2\sqrt{6}} \cdot \frac{5-2\sqrt{6}}{5-2\sqrt{6}}$$

$$= 10\left(20+5\sqrt{6}-5\sqrt{2}-8\sqrt{6}-12+4\sqrt{3}\right)$$

$$r = 80 - 50\sqrt{2} + 40\sqrt{3} - 30\sqrt{6}$$