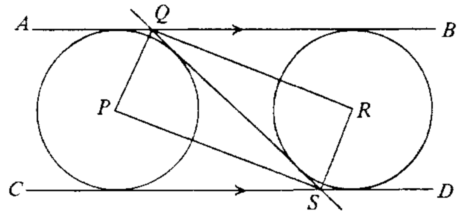
Tangent Problem 2

1971 Hong Kong Certificate of Education Examination (English) Mathematics Syllabus A Paper III Section B Q11(*b*)

Created by Francis Hung on 20080814 Last updated: 03 September 2021

In the figure, a transversal QS cuts two parallel lines AB and CD at Q and S. Circles centres P and R are drawn to touch each of the three lines as shown in the diagram. Prove that $PQRS$ is a rectangle.



Suppose AB touches the circles at E and F.

Join EP, FR and produce to meet CD at H and G.

$\angle PEF = \angle RFE = 90°$ (tangent ⊥ radius)

$\angle PHG = \angle RGH = 90°$ (int. ∠s AB // CD)

CD touches the circles at H and G (converse, tangent ⊥ radius)

Since quadrilateral $EFGH$ have 4 right angles, so it is a rectangle.

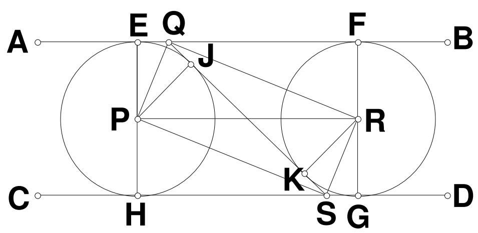
Therefore the two circles have the same radii ($EH = FG$, opp. sides of rectangle.)

Suppose the line QS touches the circles at J and K as shown.

Let $\angle PSH = \angle PSJ = \alpha$; $\angle RSK = \angle RSG = \beta$ (tangent properties)

$\alpha + \alpha + \beta + \beta = 180°$ (adj. ∠s on st. line)

$\alpha + \beta = 90°$



So $\angle PSR = \alpha + \beta = 90°$ …… (1)

$\angle KQF = \angle JSH = 2\alpha$ (alt. ∠s, AB // CD)

$\angle EQJ = \angle KSG = 2\beta$ (alt. ∠s, AB // CD)

With a similar manner, it can be easily proved that

$\angle PQR = 90°$ ……(2)

Consider ΔPQJ and ΔRSK.

$\angle PJQ = 90° = \angle RKS$ (tangent ⊥ radius)

$PJ = RK$ (same radii)

$\angle PQJ = \angle RSK = \beta$

So $\Delta PQJ \cong \Delta RSK$ (A.A.S.)

$PQ = RS$ (corr. sides ≅ Δs)

Join PR.

By (1) and (2), $\angle PSR = \angle PQR$

$PR = PR$ (common)

So $\Delta PRS \cong \Delta RPQ$ (R.H.S.)

$PS = QR$ (corr. sides ≅ Δs)

So $PQRS$ is a rectangle. (2 pairs of equal sides and one angle is 90°)

Exercise: prove that $QJ = KS$.