Tangent ⊥ radius

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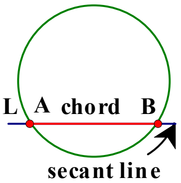
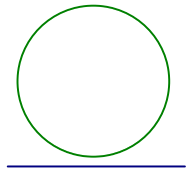
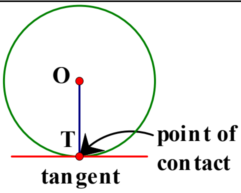
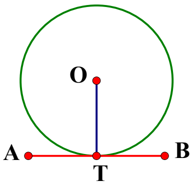
Relation between a straight line and a circle.

Case 1 The line intersects the circle at two distinct points *A* and *B*.	Case 2 The line does not intersect wthe circle	Case 3 The line touches the circle at only one point *P*.
		

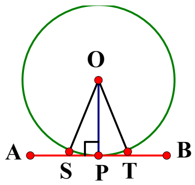
Theorem 1 Let *AB* be a tangent which touches a circle (centre at *O*) at a point *T*. Then $OT \perp AB$.



There are three cases:

case 1 $\angle ATO < 90°$, case 2 $\angle ATO > 90°$, case 3 $\angle ATO = 90°$

We shall prove that case 1 and case 2 are impossible and conclude that only case 3 $\angle ATO = 90°$ is correct.



Case 1 $\angle ATO < 90°$

Locate a point *P* on *AB* so that $OP \perp AB$. Join *OP*.

Locate a point *S* on *AP* so that $PT = PS$.

Then $\Delta OPT \cong \Delta OPS$ (S.A.S.)

$OS = OT =$ radii (cor. sides, $\cong\Delta$s)

Abbreviation: tangent ⊥ radius

$\therefore$ *S* lies on the circle.

$\because$ *S* and *T* are two distinct point on *AB* and also on the circle

$\therefore$ *AB* cuts the circle at two different points

This contradicts the fact that *AB* is a tangent.

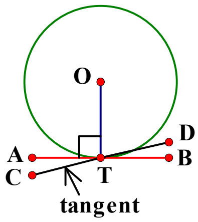
$\therefore$ Case 1 is impossible

Case 2 $\angle ATO > 90°$

Again, we can construct another $\Delta OPS \cong \Delta OPT$, which arrives at the impossible conclusion.

$\therefore$ Only case 3 is correct. $\angle ATO = 90°$

Theorem 2 Let *AB* be a line which cuts a circle (centre at *O*) at *T*. If $OT \perp AB$, then *AB* is a tangent.



Draw a tangent *CTD* through *T*

Then $\angle DTO = 90°$ (tangent ⊥ radius)

$\because \angle BTO = 90°$ and $\angle DTO = 90°$

$\therefore BT \,//\, DT$ (cor. $\angle$s eq.)

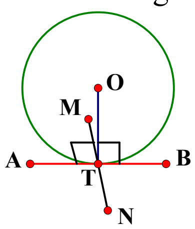
$\because$ *ATB* and *CTD* both pass through the same point *T*

$\therefore$ *ATB* overlaps with *CTD*

i.e. *AB* is a tangent to the circle

Abbreviation: **Converse, tangent ⊥ radius**

Theorem 3 Let *AB* be a tangent which touches a circle (centre at *O*) at *T*. If another line *MTN* is drawn through *T* and perpendicular to *AB*, then *MTN* must pass through the centre (*O*).



Join *OT*.

$OT \perp AB$ (tangent ⊥ radius)

$\because \angle BTO = 90°$ and $\angle BTM = 90°$

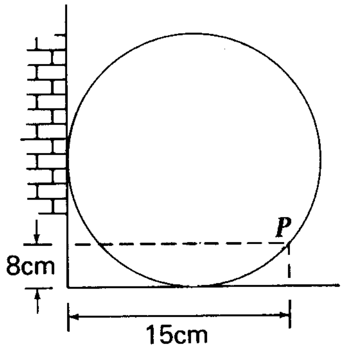
$\therefore BT \,//\, MT$ (cor. $\angle$s eq.)

$\because$ *OT* and *MTN* both pass through the same point *T*

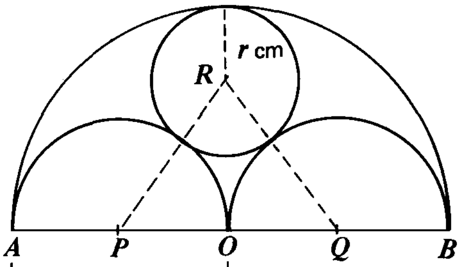
$\therefore$ *OT* overlaps with *MTN*

i.e. *MTN* must pass through the centre *O*.

1. Find the radius of the circle. [no solution]

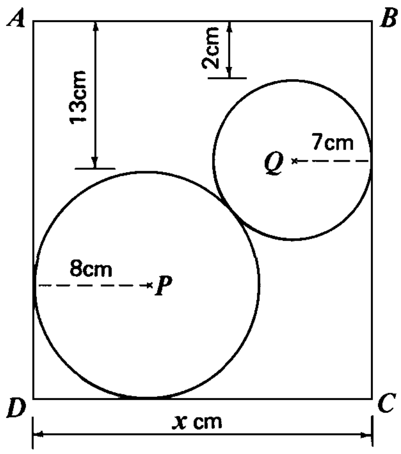


2. Find the radius of the circle. [2]

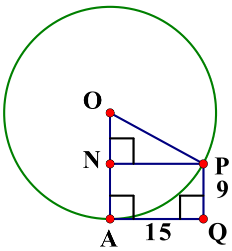


(*O* is the centre of the large semi-circle.)

3. Find x. [24]

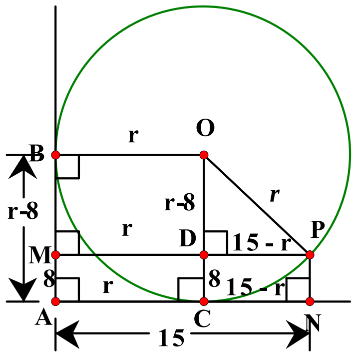


(*ABCD* is a rectangle.)

4. 

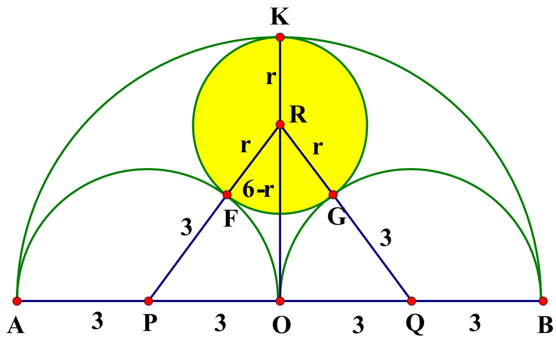
In the figure, *AQ* is a tangent which touches the circle (centre at *O*) at *A*. $\angle AQP = 90°$, $AQ = 15$, $PQ = 9$. Find the radius. [17]

1. Let O be the centre. Suppose AB is the vertical wall. AC is the horizontal ground. $\angle BAC = 90°$. Suppose the circle touches AB at B and AC at C respectively. Join OB and OC.
$OB \perp AB$, $OC \perp AC$ (tangent $\perp$ radius)
$OBMD$, $DMAC$, $PNDC$ are rectangles
$OB = DM = CA = r$, $OC = BA = r$ (opp. sides of rectangle)
$DC = MA = 8$, $MP = AN = 15$ (opp. sides of rectangle)
$CN = AN - CN = 15 - r$, $OD = OC - OD = r - 8$

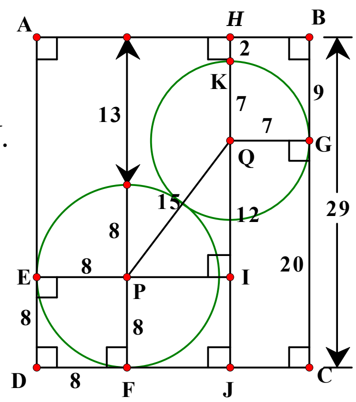


$OD \perp DP$
In ΔODP, $(r - 8)^2 + (15 - r)^2 = r^2$ (Pythagoras' theorem)
$r^2 - 16r + 64 + 225 - 30r + r^2 = r^2$
$r^2 - 46r + 289 = 0$
$r = 7.5$ or 38.5
$\because$ $r - 8 > 0$ and $15 - r > 0$
$\therefore$ Both answers are rejected
There is no solution.

2. Join PR and QR, which intersect the two smaller semi-circles at F and G respectively.
Join OR and produce it to meet the large semicircle at K.
$OK = 6$, $RF = RG = RK = r$
$\Delta POR \cong \Delta QOR$ (S.S.S.)
$\angle POR = \angle QOR = 90°$ (corr. $\angle$s, $\cong$ Δs)
In ΔPOR, $3^2 + (6 - r)^2 = (3 + r)^2$ (Pyth. theorem)
$9 + 36 - 12r + r^2 = 9 + 6r + r^2$
$r = 2$



3. $\angle A = \angle B = \angle C = \angle D = 90°$
$OE \perp AD$, $PF \perp DC$, $QG \perp BC$, $QH \perp AB$ (tangent $\perp$ radius)
Produce EP and HQ to meet at I.
HQ cuts the circle with radius 7 at K and HI produced cuts BC at J.
$AD = 13 + 8 + 8 = 29 = BC$, $IJ = PF = 8$
$QI = 29 - 2 - 7 - 8 = 12$
$PI \perp QI$
In ΔPQI, $PI^2 + 12^2 = 15^2$ (Pythagoras' theorem)
$PI = 9$
$x = BC = 8 + 9 + 7 = 24$



4. $OA \perp AQ$ (tangent $\perp$ radius)
$AQPN$ is a rectangle
$AN = QP = 9$ (opp. sides of rectangle)
$NP = 15$ (opp. sides of rectangle)
Let $OA = OP = r$
$ON = r - 9$
In ΔONP, $(r - 9)^2 + 15^2 = r^2$ (Pythagoras' theorem)
$r^2 - 18r + 306 = r^2$
$r = 17$