Solving Minimum Problem by Ptolemy's inequality

Created by Mr. Francis Hung on 20210828 Last updated: 2021-09-22

Ptolemy's Inequality

The lengths of sides and diagonals of quadrilateral *ABCD* are
$AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = p$, $BD = q$.

If *ABE* is the triangle similar (and similarly oriented) to triangle *ADC* with *AB* and *AD* as corresponding sides, express *EB* in terms of *a*, *c* and *d*, and *EC* in terms of *p*, *q* and *d*. Hence prove

(i) that $pq \le ac + bd$, and

(ii) that if the equality sign holds, *ABCD* is a cyclic quadrilateral, and conversely.

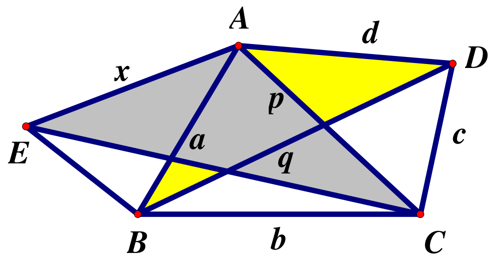
$\because \Delta ADC \sim \Delta ABE$

$\therefore \frac{EB}{a} = \frac{c}{d} \Rightarrow EB = \frac{ac}{d}$

Let $\angle BAE = \theta = \angle CAD$ (corr. ∠s. ~ ∆s), let $AE = x$

$\frac{x}{a} = \frac{p}{d}$ and $\angle EAC = \theta + \angle BAC = \angle BAD$

$\therefore \Delta EAC \sim \Delta BAD$ (ratio of 2 sides, included ∠)



$\frac{EC}{q} = \frac{p}{d}$ (ratio of sides, ~∆s)

$EC = \frac{pq}{d}$

(i) In ΔBCE, $EB + BC \ge EC$ (triangle inequality)

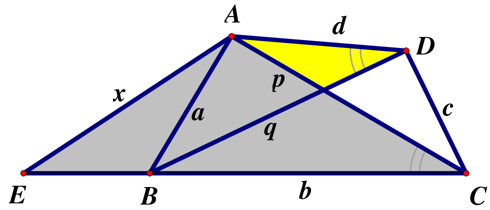
$\frac{ac}{d} + b \ge \frac{pq}{d}$

$ac + bd \ge pq$

(ii) If equality holds, then *EBC* is a straight line.

$\because \Delta EAC \sim \Delta BAD \therefore \angle ACE = \angle ADB$

ABCD is a cyclic quadrilateral
(converse, ∠s in the same segment)



Converse, if *ABCD* is a cyclic quadrilateral

$\angle ACB = \angle ADB$ (∠s in the same segment)

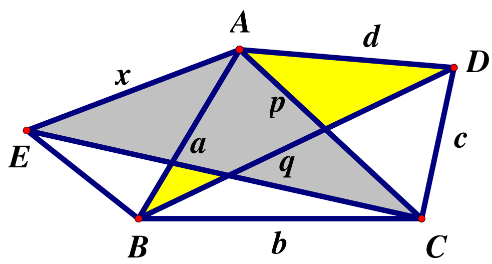
$\because \Delta EAC \sim \Delta BAD \therefore \angle ACE = \angle ADB$

$\therefore \angle ACB = \angle ADB = \angle ACE$

EBC is a straight line

$EB + BC = EC$

$ac + bd = pq$



Given $x^2 + y^2 = 4$, find the minimum value of $3\sqrt{5-2x}+\sqrt{13-6y}$.

$x^2 + y^2 = 4$

Let $T = 3\sqrt{5-2x}+\sqrt{13-6y} = \sqrt{45-18x}+\sqrt{13-6y}$

$= 3\sqrt{1-2x+x^2+4-x^2}+\sqrt{9-6y+y^2+4-y^2}$

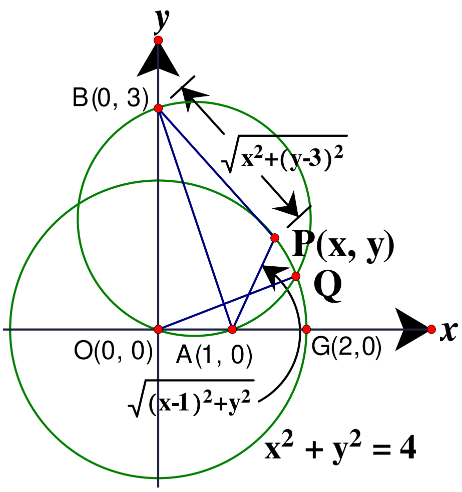
$= 3\sqrt{(x-1)^2+y^2}+\sqrt{x^2+(y-3)^2}$

Consider the following problem.

Let $P(x, y)$ be any point on the circle $x^2 + y^2 = 4$ (centre at $O(0, 0)$, and radius = 2).

$A(1, 0)$, $B(0, 3)$ be two given points.

$T = 3\sqrt{(x-1)^2+y^2}+\sqrt{x^2+(y-3)^2} = 3AP + BP$ as shown in the figure.



By Ptolemy's inequality, $OB \times AP + OA\times BP \geq AB\times OP$,

with equality holds when $OAPB$ is a cyclic quadrilateral.

i.e. $3AP + BP \geq \sqrt{10}\times OP$

In this case, $\angle AOB = 90°$.

$\therefore$ AB is the diameter of the circle passing through A, O, B .

Let Q be the point of intersection of this circle with the circle $x^2 + y^2 = 4$ in the first quadrant.

When $OAPB$ is a cyclic quadrilateral, $P = Q$

Minimum value of $T = 3AP + BP = \sqrt{10}\times OQ = 2\sqrt{10}$