Euler Line and 9-point circle

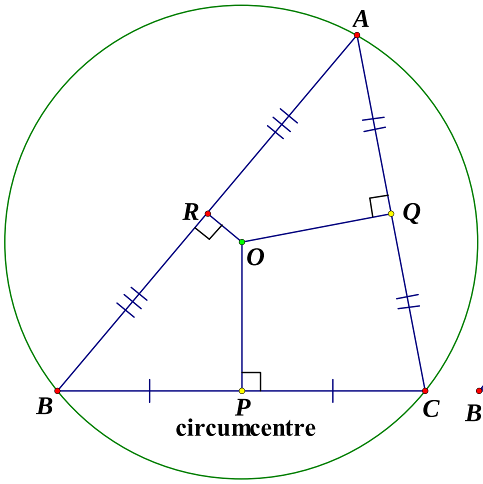
Created by Francis Hung on 13-11-2017 Last updated: 08 November 2022

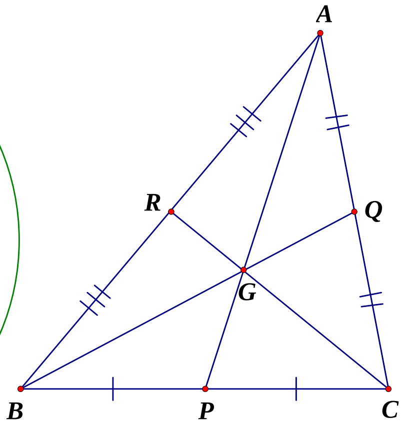
Euler line

The circumcentre O, centriod G and the orthocentre H are collinear.

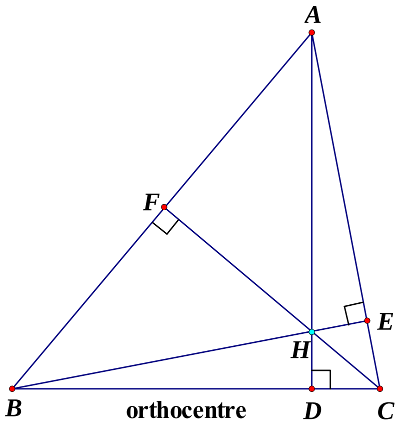
Furthermore, G divides OH in the ratio 1 : 2.

Recall the definition of circumcentre, centriod and orthocentre:

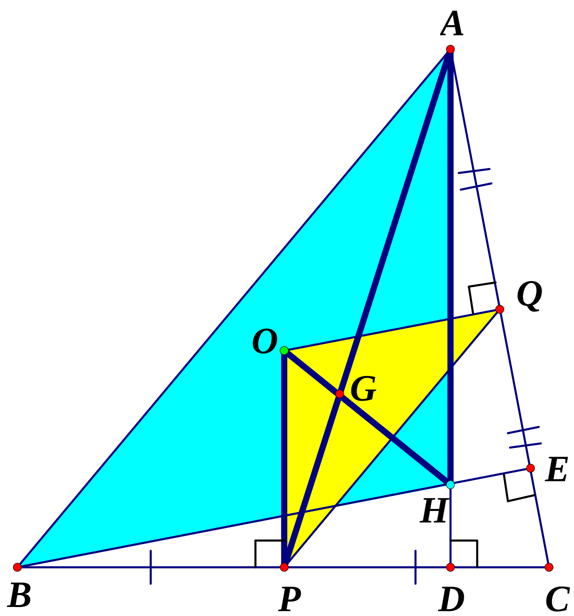




centriod



In the figure, $BP = PC$, $AQ = QC$.
The perpendicular bisectors QO and PO meet at O.
The altitudes AD and BE meet at H.
OH meets the median AP at G.
Try to show that G = centriod and $OG : GH = 1 : 2$.



$OP \parallel AH$, $OQ \parallel BH$ (altitudes)
$PQ \parallel AB$ (mid point theorem)
$\therefore \Delta OPQ \sim \Delta HAB$ (3 pairs of // lines)
$PQ = \frac{1}{2}AB$ (mid point theorem)
$\therefore OP = \frac{1}{2}AH \cdots (*)$ (corr. sides, ~Δs)
$\angle AGH = \angle PGO$ (vert. opp. ∠s)
$\angle GAH = \angle GPO$ (alt. ∠s, $AH \parallel OP$)
$\angle GHA = \angle GOP$ (alt. ∠s, $AH \parallel OP$)
$\Delta AGH \sim \Delta PGO$ (equiangular)
$OG : GH = OP : AH$ (corr. sides, ~ Δs)
$= 1 : 2$ (proved in (*))
$PG : GA = OP : AH$ (corr. sides, ~ Δs)
$= 1 : 2$
$\therefore$ G is the centriod.

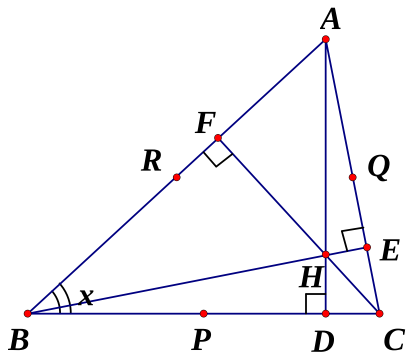
The theorem is proved.

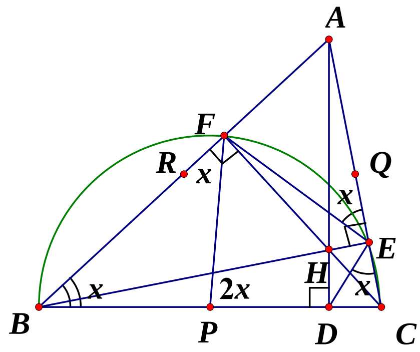
The line segment joining *O* and *H* is called the Euler line.

9-point circle

Theorem 1

In ΔABC, AD, BE and CF are the altitudes which intersects at the orthocentre H.
P, Q and R are mid points of BC, CA and AB respectively. Then D, E, F and P are concyclic.



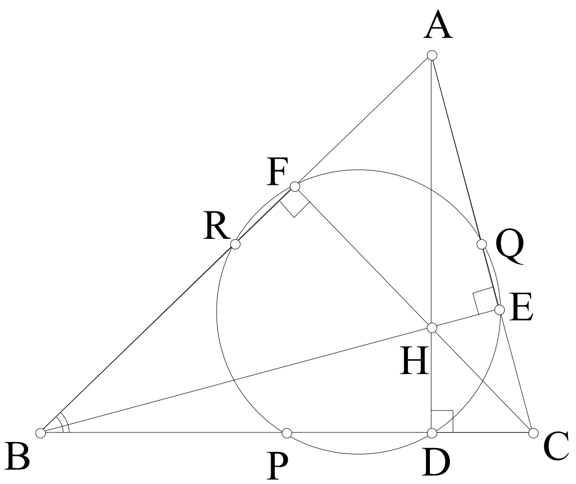


$\because \angle BFC = \angle BEC = 90°$	(Given BE and CF are altitudes)
$\therefore B, C, E, F$ are concyclic	(Converse $\angle$s in the same segment)
Let $\angle ABC = x$.	
$\angle AEF = x$	(ext $\angle$, cyclic quad.)
$\because \angle AEB = \angle ADB = 90°$	(Given BE and AD are altitudes)
$\therefore A, B, D, E$ are concyclic	(Converse $\angle$s in the same segment)
$\angle CED = x$	(ext $\angle$, cyclic quad.)
$\angle DEF = 180° - 2x$	(adj. $\angle$ on st. line)
BC = diameter of the circle $BFEC$.	(Converse, $\angle$ in semi-circle)

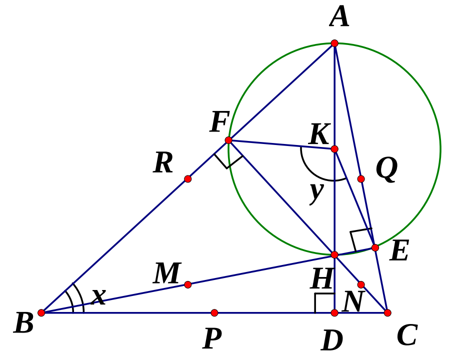
$\because P$ = mid point of the diameter BC $\therefore P$ = centre of the circle $BCEF$.

$BP = PF$ = radii	
$\angle PFB = x$	(base $\angle$s, isos. Δ)
$\angle FPD = \angle PFB + \angle PBF = 2x$	(ext. $\angle$ of Δ)
$\angle FPD + \angle DEF = 2x + 180° - 2x = 180°$	
$\therefore EFPD$ is a cyclic quadrilateral.	(opp. $\angle$s supp.)

The theorem is proved.
Using this theorem, we can draw a circle which passes through P, D, E, Q, F, R.



Theorem 2



In ΔABC, AD, BE and CF are the altitudes which intersects at the orthocentre H.

K, M, N are mid points of AH, BH and CH respectively.

Then D, E, K and F are concyclic.

Join FK, EK. Let $\angle FKE = y$.

$\angle AEH + \angle AFH = 180°$

A, E, H, F are concyclic (opp. $\angle$s supp.)

$\therefore AH$ = diameter of the circle $AEHF$.

(Converse, $\angle$ in semi-circle)

$\because$ K is the mid point of AH

$\therefore$ K = centre of the circle

$\angle BAC = \frac{y}{2}$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$\because$ $\angle AFC = \angle ADC = 90°$

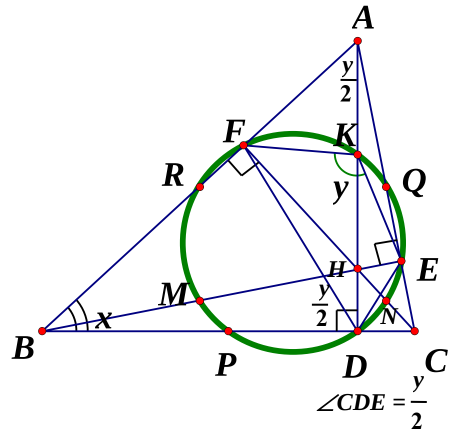
$\therefore$ A, C, D, F are concyclic (converse $\angle$s in same seg.)

$\angle BDF = \frac{y}{2}$ (ext. $\angle$, cyclic quad.)

$\because$ $\angle AEB = \angle ADB = 90°$

$\therefore$ A, B, D, E are concyclic (converse $\angle$s in same seg.)

$\angle CDE = \frac{y}{2}$ (ext. $\angle$, cyclic quad.)



$\angle EDF = 180° - 2\times\frac{y}{2} = 180° - y$ (adj. $\angle$s on st. line)

$\angle EDF + \angle EKF = 180° - y + y = 180°$

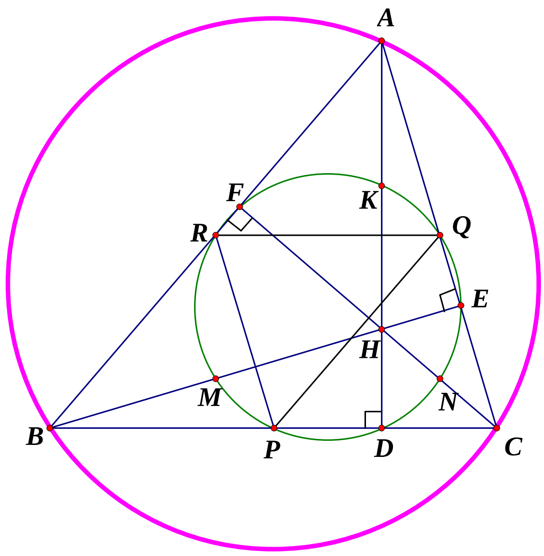
$\therefore$ D, E, F and K are concyclic (opp. $\angle$s supp.)

The theorem is proved.

Using this theorem, we can draw a circle which passes through D, N, E, K, F, M:

From **theorem 1** and **theorem 2**, the circle passes through D, E, F, K, M, N, P, Q, R is called the **nine-point circle**.

The radius of 9-point circle half circumradius



Let $BC = a$, $CA = b$, $AB = c$.

$\because$ The 9-point circle passes through the mid-points of *BC*, *CA* and *AB* respectively.

$\therefore\ QR = \frac{a}{2}$, $PR = \frac{b}{2}$, $PQ = \frac{c}{2}$ and *RQ* // *BC*, *RP* // *AC*, *QP* // *AC* (mid-point theorem)

$\angle QPR = A$, $\angle PQR = B$, $\angle PRQ = C$.

Let the radii of the 9-point circle and the circumcircle be *r* and *s* respectively.

Then by sine formula,

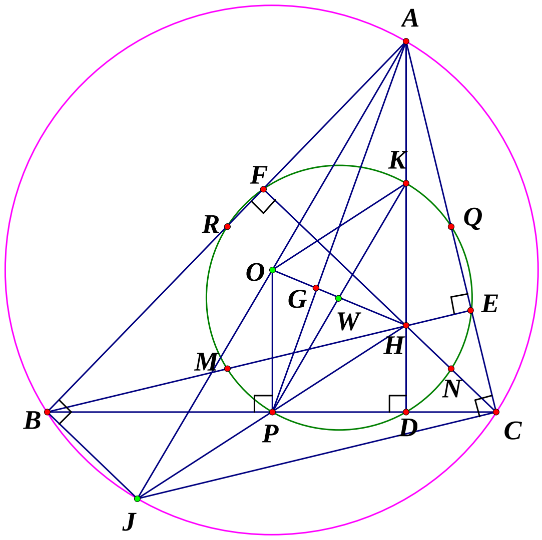
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{s}{2} \quad \cdots\cdots (1), \quad \frac{\frac{a}{2}}{\sin A} = \frac{\frac{b}{2}}{\sin B} = \frac{\frac{c}{2}}{\sin C} = \frac{r}{2} \quad \cdots\cdots (2)$$

(1)÷(2): $2 = \frac{s}{r}$

$\therefore\ r = \frac{s}{2}$

The result is proved.

The centre (W) of the 9-point circle lies mid-way between the circumcentre (O) and the orthocentre (H)



In ΔABC, P, Q, R are mid points of BC, CA and AB respectively. AD, BE, CF are the altitudes. H is the orthocentre. K, M, N are mid points of AH, BH, CH respectively. O is the circumcentre. Join AO and produce AO to the other end J of the circumscribed circle. Join BJ, CJ, OP, PK, OK, PH, PJ. Suppose W is the mid-point of OH.

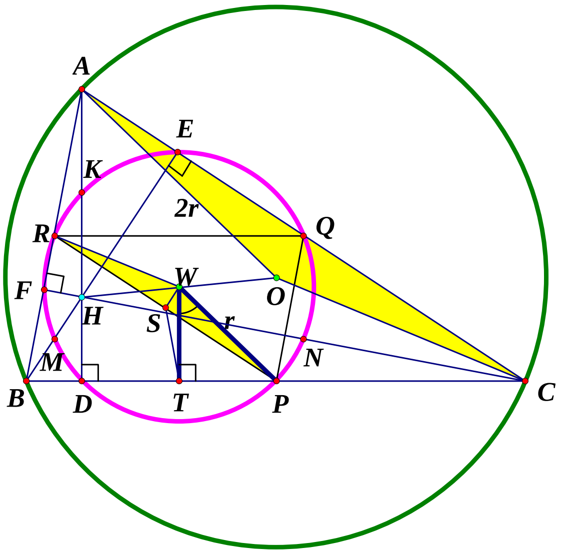
$\angle ACJ = 90°$	($\angle$ in semi-circle)
$\angle AEB = 90°$	($\because$ BE = altitude)
$\therefore$ BE // JC	(corr. $\angle$ eq.)
$\angle ABJ = 90°$	($\angle$ in semi-circle)
$\angle BFC = 90°$	($\because$ CF = altitude)
$\therefore$ JB // CF	(int. $\angle$ supp.)
$\therefore$ $BJCH$ is a parallelogram	(2 pairs of //-lines)
$BP = PC$ and $JP = PH$ $\cdots\cdots$ (1)	(diagonals of //-gram)
J, P, H are collinear	
$\because$ O = mid point of AJ and K = mid point of AH	
$\therefore$ $OK = \frac{1}{2} JH$ and OK // JH	(mid point theorem)
$\therefore$ $OK = PH = JP$ and OK // JH	by (1)
$\therefore$ $OPHK$ is a parallelogram	(opp. sides are eq. and parallel)
$\because$ The nine-point circle passes through P, K, D and $\angle PDK = 90°$	
$\therefore$ PK is the diameter of the nine-point circle	(Converse, $\angle$ in semi-circle)
$PW = WK$ and $OW = WH$	(diagonals of //-gram)
W is the centre of the nine-point circle.	($\because$ PW=WK= radius of the 9-point circle)
$\because$ W = mid point of OH and K = mid point of AH	
$\therefore$ $WK = \frac{1}{2} OA$	(mid point theorem)

$\therefore$ The radius of the nine point circle is half of the radius of the circumscribed circle.

The theorem is proved.

Together with the centriod G, we have $OG : GW : WH = 2 : 1 : 3$.

Trilinear coordinates of 9-point centre *W*



In ΔABC, P, Q, R are mid points of BC, CA and AB respectively. AD, BE, CF are the altitudes. H is the orthocentre. K, M, N are mid points of AH, BH, CH respectively. O is the circumcentre. W is the centre of the 9-point circle. S is the mid-point of PR, T is the mid-point of PD.

$WP = WR = r$, $OA = OC = 2r$	(radii)
$2RP = AC$	(mid-point theorem)
$\Delta PQW \sim \Delta ACO$	(3 sides proportional)
$\angle AOC = 2\angle B$	($\angle$ at centre twice $\angle$ at $\odot^{ce}$)
$\angle PWR = \angle AOC = 2\angle B$	(corr. $\angle$s ~Δs)
$\Delta PSW \cong \Delta RSW$	(S.S.S.)
$\angle PWS = \angle RWS = \angle B$	(corr. $\angle$s $\cong\Delta$s)
$WS \perp PR$ and $WT \perp DP$	(line joing centre to mid-point of chord $\perp$ chord)
W, S, T, P are concyclic	(converse, $\angle$s in the same segment)
$\angle SWT = \angle SPT$	($\angle$s in the same segment)
$= \angle C$	(P and R are mid-points of BC and AB, mid-point thm)

$\therefore \angle PWT = \angle PWS - \angle SWT = \angle B - \angle C$

The distance from W to $BC = r \cos \angle PWT = r \cos(B - C)$

Similarly, the distances from W to $AC = r \cos(C - A)$ and W to $AB = r \cos(A - B)$

Trilinear coordinates of W $= r \cos(B - C) : r \cos(C - A) : r \cos(A - B)$

$$= \mathbf{\cos(B - C) : \cos(C - A) : \cos(A - B)}$$