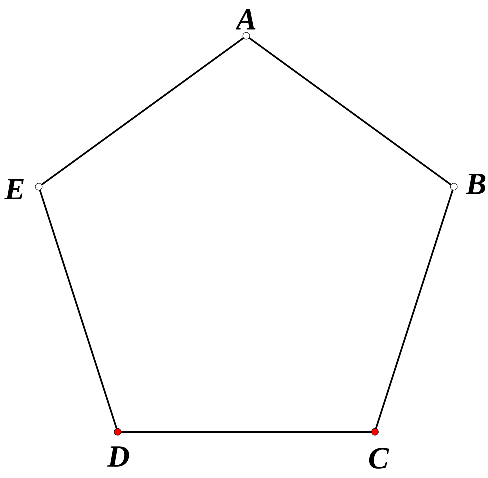
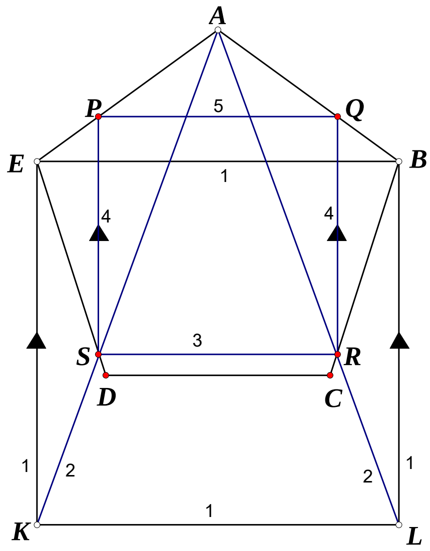
Given a regular pentagon *ABCDE*. To construct an inscribed square *PQRS* in *ABCDE* so that *RS* // *CD*.

Created by Mr. Francis Hung on 2017-02-07. Last updated: 2023-07-03.





Construction steps:

(1) Join BE. Construct a square $BEKL$. (KL and DC lie on the same side of EB.)

(2) Join AK, cutting DE at S. Join AL, cutting BC at R.

(3) Join SR. Draw SP // KE, cutting AE at P. Draw RQ // LB, cutting AB at Q.

(4) Join PQ. Then $PQRS$ is the required inscribed square.

Proof: $AE = AB$ (sides of the regular pentagon)

$EK = BL$ (opp. sides of a square)

$\angle BAE = 108°$ ($\angle$ sum of polygon)

$\angle AEB = \angle ABE = (180° - 108°)\div 2 = 36°$ ($\angle$ sum of Δ, base $\angle$s isos. Δ)

$\angle KEB = \angle LBE = 90°$ (property of a square)

$\angle AEK = 36° + 90° = 126° = \angle ABL$

$\Delta AEK \cong \Delta ABL$ (S.A.S.)

$AK = AL$ (corr. sides $\cong \Delta$s)

$AE = AB$ (proved)

$\angle EAS = \angle BAR$ (corr. $\angle$s $\cong \Delta$s)

$\angle AES = \angle ABR = 108°$ ($\angle$ sum of polygon)

$\Delta AES \cong \Delta ABR$ (A.A.S.)

$AS = AR$ (corr. sides $\cong \Delta$s)

$\therefore \quad \dfrac{AS}{AK} = \dfrac{AR}{AL}$

$\angle SAR = \angle KAL$ (common $\angle$s)

$\Delta ASR \sim \Delta AKL$ (2 sides proportional, included $\angle$)

$\angle ASR = \angle AKL$ (corr. $\angle$s $\sim \Delta$s)

SR // KL (corr. $\angle$s eq.)

$\angle PSR = \angle PSA + \angle ASR$

$= \angle EKA + \angle AKL$ (corr. $\angle$s, KE // SB, KL // SR)

$= \angle EKL = 90°$ (property of a square)

PS // EK // BL // QR (transitive property of // lines)

$\angle QRS = 180° - 90° = 90°$ (int. $\angle$s, BD // QR)

$\Delta APS \sim \Delta AEK$ and $\Delta AQR \sim \Delta ABL$ (equiangular)

$\dfrac{PS}{EK} = \dfrac{AS}{AK}$ (corr. sides $\sim \Delta$s)

$= \dfrac{AR}{AL} = \dfrac{SR}{KL}$ (corr. sides $\sim \Delta$s)

$= \dfrac{QR}{BL}$ (corr. sides ~ Δs)

$\because EK = BL$ and $EK = KL$ (property of a square)

$\therefore PS = SR = QR$

$PQRS$ is a //-gram (opp. sides are eq. and //)

$PQRS$ is a square (adj. sides are eq.)

Let $AB = BC = CD = DE = 2a$ and let $\theta = 36°$

$5\theta = 180° \Rightarrow 3\theta = 180° - 2\theta$

$\sin 3\theta = \sin(180° - 2\theta) = \sin 2\theta$

$3 \sin \theta - 4 \sin^3 \theta = 2 \sin \theta \cos \theta$

$3 - 4 \sin^2 \theta = 2 \cos \theta$

$3 - 4(1 - \cos^2 \theta) = 2 \cos \theta$

$4 \cos \theta - 2 \cos \theta - 1 = 0$

$\cos \theta = \dfrac{1+\sqrt{5}}{4}$ or $\dfrac{1-\sqrt{5}}{4}$ (rejected)

In ΔABE, $AB = AE = 2a$, $\angle BAE = 108°$ (∠ sum of polygon)

$\angle ABE = \angle AEB = (180° - 108°)\div 2 = 36°$ (∠ sum of Δ, base ∠s isos. Δ)

$BE = 2\, AE \cos \angle\, AEB = 4a \cos 36° = \left(1+\sqrt{5}\right)a = BL = KL = EK$

$\angle AES = 108°$ (∠ sum of polygon)

$\angle BES = 108° - 36° = 72°$

$\angle KES = 90° - 72° = 18°$

Apply sine rule on ΔAES and ΔKES. Let $\angle ASE = \alpha$, then $\angle KSE = 180° - \alpha$ (adj. ∠s on st. line)

$\dfrac{AS}{\sin 108^\circ} = \dfrac{2a}{\sin\alpha}$ …… (1) $\dfrac{SK}{\sin 18^\circ} = \dfrac{\left(1+\sqrt{5}\right)a}{\sin\left(180^\circ - \alpha\right)}$ …… (2)

Using the fact that $\sin(180° - \alpha) = \sin \alpha$, $\sin 108° = \cos 18°$ and (2)÷(1):

$\dfrac{SK}{AS}\cdot\dfrac{\cos 18^\circ}{\sin 18^\circ} = \dfrac{\left(1+\sqrt{5}\right)a}{2a}$

$\dfrac{SK}{AS} = \dfrac{1+\sqrt{5}}{2}\cdot\tan 18^\circ$

$\dfrac{AK}{AS} = \dfrac{AS+SK}{AS} = 1 + \dfrac{SK}{AS} = 1 + \dfrac{1+\sqrt{5}}{2}\cdot\tan 18^\circ = \dfrac{2+\left(1+\sqrt{5}\right)\tan 18^\circ}{2}$

$\dfrac{AS}{AK} = \dfrac{2}{2+\left(1+\sqrt{5}\right)\tan 18^\circ}$

$\dfrac{PS}{EK} = \dfrac{AS}{AK} = \dfrac{2}{2+\left(1+\sqrt{5}\right)\tan 18^\circ}$

PS = the length of a side of the square $= \dfrac{2\left(1+\sqrt{5}\right)a}{2+\left(1+\sqrt{5}\right)\tan 18^\circ}$

Let $\beta = 18°$, then $5\beta = 90°$, $3\beta = 90° - 2\beta$, let $\tan \beta = t$

$\tan 3\beta = \tan(90° - 2\beta) = \cot 2\beta$

$\dfrac{3t - t^3}{1 - 3t^2} = \dfrac{1 - t^2}{2t}$

$6t^2 - 2t^4 = 1 - t^2 - 3t^2 + 3t^4$

$5t^4 - 10t^2 + 1 = 0$

$t^2 = \dfrac{5+\sqrt{20}}{5}$ (rejected) or $\dfrac{5-\sqrt{20}}{5} = \dfrac{5-2\sqrt{5}}{5}$

$\tan 18^\circ = \sqrt{\frac{5-2\sqrt{5}}{5}}$

$$PS = \frac{2\left(1+\sqrt{5}\right)a}{2+\left(1+\sqrt{5}\right)\tan 18^\circ}$$

$$= \frac{2a}{\frac{2}{1+\sqrt{5}}+\sqrt{\frac{5-2\sqrt{5}}{5}}}$$

$$= \frac{2a}{\frac{\sqrt{5}-1}{2}+\sqrt{\frac{5-2\sqrt{5}}{5}}}$$

$$= \frac{4\sqrt{5}a}{5-\sqrt{5}+2\sqrt{5-2\sqrt{5}}}$$

$$\approx 2.12a$$