Given the 3 altitudes h_a, h_b, h_c of a triangle, to construct the triangle *ABC*

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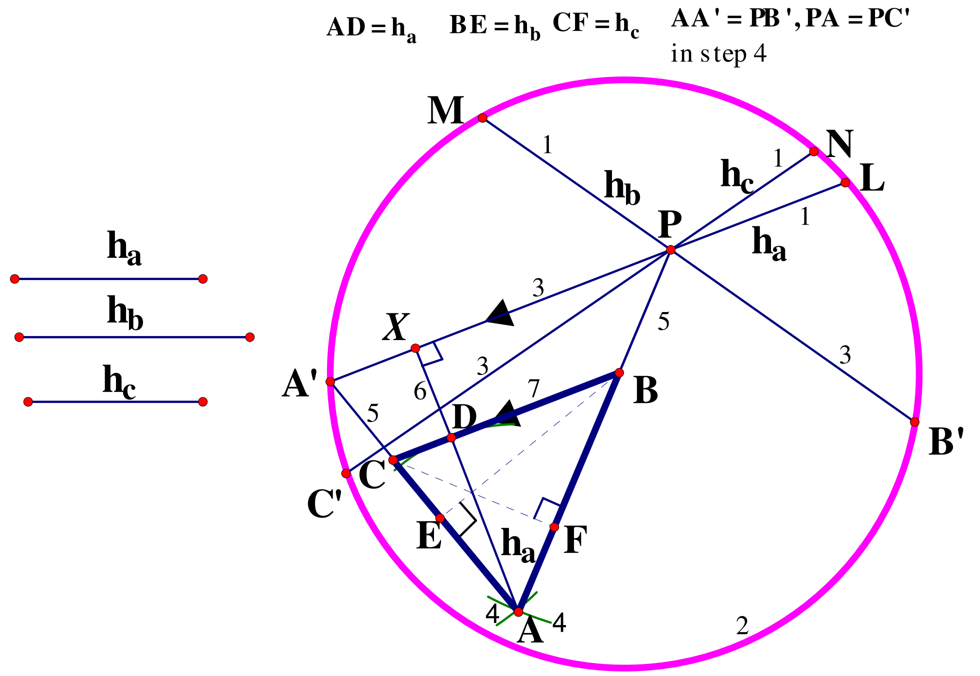

Construction steps

(1) Let P be any point. Draw the line segments $PL = h_a$, $PM = h_b$, $PN = h_c$ such that L, M, N are not collinear.

(2) Construct the circumscribed circle through L, M and N.

(3) Extend LP, MP and NP to meet the circle again at A', B' and C' respectively.

(4) Use P as centre, PC' as radius to draw an arc; use A' as centre, PB' as radius to draw another arc. The two arcs intersect at A.

(5) Join AA', AP.

(6) Draw the altitude $AX \perp A'P$. Locate a point D on AX so that $AD = h_a$.

(7) Draw a line segment BDC parallel to PA', cutting AP and AA' at B and C respectively.

Then ΔABC is the required triangle.

Proof: By intersecting chords theorem, $h_a \times PA' = h_b \times PB' = h_c \times PC'$ ······ (1)

By step (4), $\Delta APA'$ have lengths of sides equal to PA', PB' and PC'.
By calculating the area of $\Delta APA'$ in different ways,
Height through $A \times PA'$ = Height through $P \times PB'$ = Height through $A' \times PC'$ ······ (2)

By step (6), $\Delta APA' \sim \Delta ABC$ (equiangular) and the altitude $AD = h_a$.
$AP : PA' : AA' = AB : BC : CA$ (corr. sides, ~Δs)
Let the heights of ΔABC be AD, BE and CF be as shown in the figure.
Then $AD : BE : CF = AX$: altitude of $\Delta APA'$ through P : altitude of $\Delta APA'$ through A' ······ (3)

Compare (1), (2), (3), we have
$AD : BE : CF = h_a : h_b : h_c$
Use the fact that $AD = h_a$, we have $BE = h_b$, $CF = h_c$.
The proof is completed.