Given three angle bisectors. To construct ΔABC.

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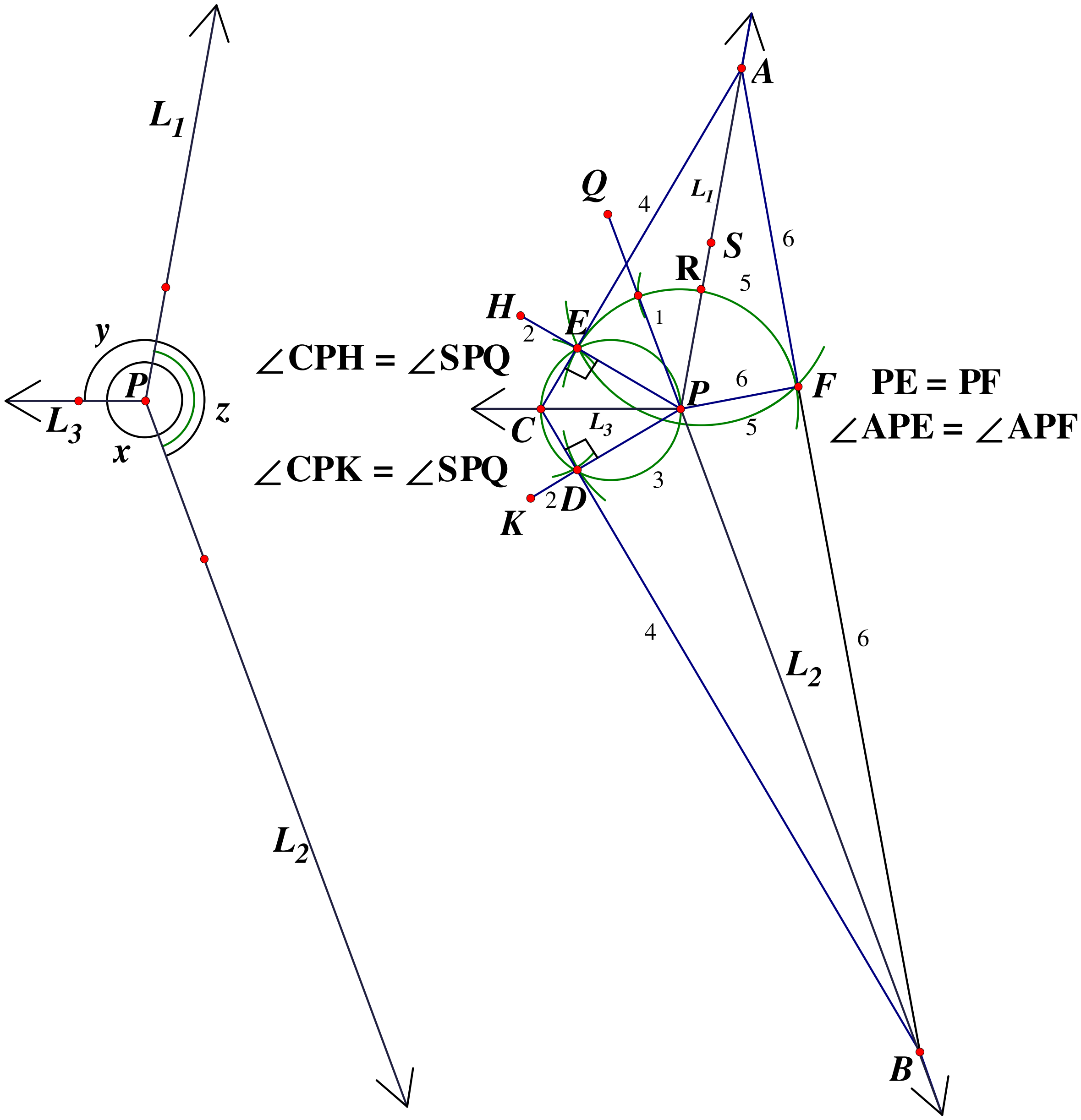
Given a point P. 3 rays L_1, L_2, L_3 are drawn from P. The angles between L_2 and L_3, L_3 and L_1, L_1 and L_2 are x, y and z respectively, where x, y and z are obtuse angles. To construct ΔABC on L_1, L_2 and L_3 so that L_1, L_2 and L_3 are the angle bisectors of ΔABC.



Construction steps

(1) Extend L_2 backwards to PQ.

(2) Let S and C be any points on L_1 and L_3 respectively.
Copy $\angle SPQ$ on PC so that $\angle CPH = \angle SPQ$, $\angle CPK = \angle SPQ$.

(3) Draw a circle with CP as diameter, cutting PH at E and PK at D respectively.

(4) Join CE and produce it to cut L_1 at A. Join CD and produce it to cut L_2 at B.

(5) Use P as centre, PE as radius to draw an arc, cutting L_1 at R.
Use R as centre, RE as radius to draw another arc. The two arcs intersect again at F.

(6) Join PF, AF and FB.

Then ΔABC is the required triangle.

Proof: $x + y + z = 360° \cdots\cdots (1)$ ($\angle$s at a point)

$\angle APQ = 180° - z$ (adj. $\angle$s on st. line L_2)

$\angle CPH = \angle CPK = 180° - z$ (by construction)

$\angle CEP = \angle CDP = 90°$ ($\angle$ in semi-circle)

$\angle PCE = \angle PCD = 90° - (180° - z) = z - 90°$ ($\angle$ sum of Δ)

$\therefore$ CP is the angle bisector of $\angle ACB$.

$\Delta PCE \cong \Delta PCD$ (A.A.S.)

By steps (5) and (6), $\angle APE = \angle APF$

$= y - \angle CPE$

$= y - (180° - z)$

$= y + z - 180°$

$= 360° - x - 180°$ (by (1))

$= 180° - x$

$PE = PF$ (by step (5))

$PA = PA$ (common side)

$\therefore$ $\Delta APF \cong \Delta APE$ (S.A.S.)

$\angle FAP = \angle EAP$ (corr. $\angle$s $\cong$ Δs)

$\therefore$ AP is the angle bisector of $\angle BAC$.

$\angle BPD = x - \angle CPD = x - (180° - z) = x + z - 180°$

$= 360° - y - 180° = 180° - y$

$\angle BPF = z - \angle APF = z - (180° - x) = x + z - 180°$

$= 360° - y - 180° = 180° - y$

$\therefore$ $\angle BPD = \angle BPF$

$PD = PE = PF$ (corr. sides $\cong$ Δs and step 5)

$BP = BP$ (common side)

$\Delta BPD \cong \Delta BPF$ (S.A.S.)

$\angle BFP = \angle BDP = 90°$ (corr. $\angle$s $\cong$ Δs)

$\therefore$ A, F, B are collinear ($\angle AFP + \angle BFP = 180°$)

$\angle FBP = \angle DBP$ (corr. $\angle$s $\cong$ Δs)

$\therefore$ BP is the angle bisector of $\angle ABC$.

The proof is completed.