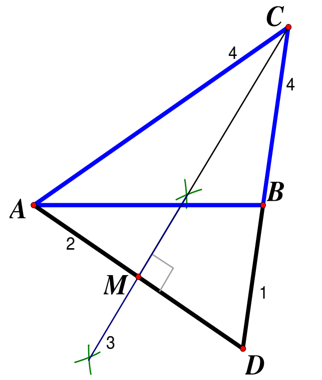
香港數學競賽 2011 初賽(幾何作圖)第 2 題

The figure shows a line segment AB. D is a fixed point such that A, B, D are not collinear. Construct a triangle ABC so that C, B and D are collinear and $AC - BC = BD$.

如圖所示為一綫段 AB。D 為一固定點，且 A、B、D 不共綫。試作ΔABC，使得 C、B 及 D 共綫，及 $AC - BC = BD$。

(1) Join DB and produce it further.
(2) Join AD.
(3) Construct the perpendicular bisector of AD, cutting DB produced at C. Let M be the mid-point of AD.
(4) Join AC, BC.



ΔABC is the required triangle.

Proof: $\Delta ACM \cong \Delta DCM$ (S.A.S.)

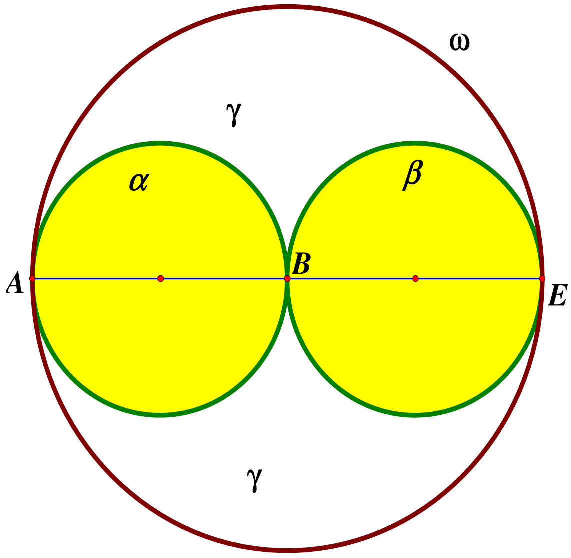
$AC = DC$ (corr. sides, $\cong \Delta$s)

$BC = DC - BD = AC - BD$

$\therefore BD = AC - BC$.

Discussion: There are some positions on the plane for which ΔABC is not constructible.

Use B as centre BA as radius to draw a circle.
Let AE be the diameter of this circle. $AB = BE$.
Draw 2 circles with AB, BE as diameters.
Let the region (including the boundary) bounded by the circles with AB, BE as diameters be α and β respectively.
Let the region inside the great circle with AE as diameter but not in α and β be γ.
Let the region on or outside the great circle be ω.



When D lies on ω, then $BD \ge AB$

So $AC - BC \ge AB$

$AC \ge AB + BC$

This inequality violates the triangle law:
The sum of 2 sides of a triangle is larger than the third side.

i.e. $AC \ge AB + BC > AC$, which is false.

$\therefore$ **When D lies on or outside the circle, ΔABC is not constructible.**

When D lies on α,

$\Delta AC_1M \cong \Delta DC_1M$ (S.A.S.)

$\angle C_1AM = \angle C_1DM$ (corr. $\angle$s, $\cong \Delta$s)

$\angle ADC_1 = \angle BAD + \angle ABD$ (ext. $\angle$ of ΔABD)

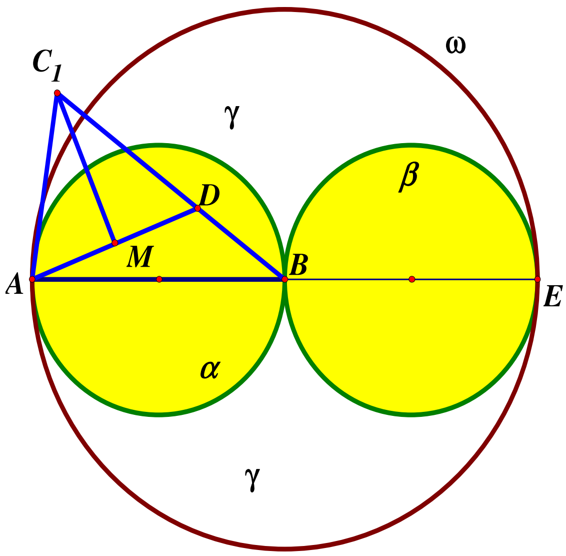
$> \angle ABD$

$\therefore \angle C_1AM = \angle ADC_1 > \angle ABD$

$\angle C_1AB > \angle ABD$

$AC_1 < BC_1$ (greater sides opp. greater $\angle$s)

The contradicts to the fact that $BD = AC_1 - BC_1$ because $AC_1 = BD + BC_1 > BC_1$



In this case the steps are as follows:
(1) Join DB and produce it further.
(2) Use B as centre, BD as radius to draw a circle, cutting DB produced at F.
(3) Join AF.

(4) Construct the perpendicular bisector of AF, cutting DB at C_2. Let M_2 be the mid-point of AF.

(5) Join AC_2, BC_2.

ΔABC_2 is the required triangle.

$\angle ABC_2 = \angle BAF + \angle AFB$ (ext. $\angle$ of ΔABF)

$> \angle AFB$

$= \angle FAC_2$ (corr. $\angle$s, $\cong \Delta$s)

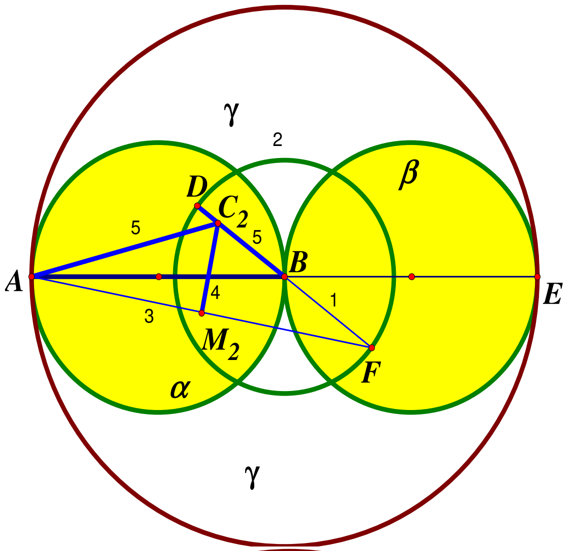
$> \angle BAC_2$

$\therefore AC_2 > BC_2$ (greater sides opp. greater $\angle$s)

The triangle ABC is constructible.

When D lies on β, relabel D as F and F as D, the construction steps are shown above.

There is only one possible triangle.



When D lies on the boundary of α,

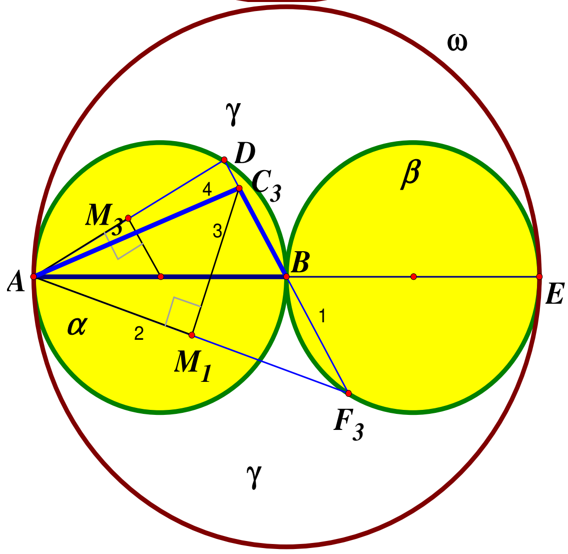
$\angle ADB = 90°$ ($\angle$ in semi-circle)

The perpendicular bisector of AD is parallel to DB. $\therefore$ It will not intersect DB produced.

Suppose DB produced intersects the boundary of β at F_3. Then the perpendicular bisector of AF_3 intersects DB at C_3.

ΔABC_3 is the required triangle.

Proof: omitted.



When D lies on γ,

Use B as centre, BD as radius to draw a circle, cutting DB produced at E on γ.

The perpendicular bisector of AD cuts DB produced at C_4. The perpendicular bisector of AE cut BD at C_5.

$\therefore$ **We can construct 2 possible triangles** ΔABC_4 and ΔABC_5 satisfying the given conditions.